UM 101 HOMEWORK ASSIGNMENT 4

Posted on November 10, 2022
(NOT FOR SUBMISSION)

- These problems are for self-study.
- Some of these problems will be discussed at the next tutorial.
- A 15-minute quiz consisting of one problem from this assignment will be conducted at the end of the tutorial section.

You might find the following inequalities useful. For fun(?), try to prove these on your own.

(i) Given $n \in \mathbb{N}$ and $x > 0$, $(1+x)^n \geq nx$.

(ii) Given $n \in \mathbb{N}$ and $x > 0$, $(1+x)^n \geq \dfrac{n(n-1)}{2}x^2$.

(iii) The AM-GM inequality. Given $x, y > 0$, $\dfrac{x+y}{2} \geq \sqrt{xy}$.

Problem 1. Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$ such that for some $N \in \mathbb{N}$, $0 \leq a_n \leq b_n$ for all $n \neq N$. Convince yourself that if $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$. Using this fact, prove the following statements (you are not allowed to use logarithms for these proofs).

(a) For any $r > 0$, $\lim\limits_{n\to\infty} \sqrt[n]{r} = 1$.

(b) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$.

Hint. In both cases, set $x = a_n - 1$ and apply one of the inequalities listed above.

Problem 2. (a) Show that the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges. The mathematical constant e is defined as the sum of this series.

(b) **Bonus (not for quiz).** Complete the following steps to show that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ is irrational, i.e., e cannot be written as p/q for any $p \in \mathbb{Z}$ and $q \in \mathbb{P}$.

(i) Let $s_n = 1 + \frac{1}{1!} + \cdots + \frac{1}{n!}$, $n \in \mathbb{N}$. Show that, for all $n \in \mathbb{P}$,

$$0 < e - s_n < \frac{1}{n!n}. \tag{1}$$

(ii) Suppose $e = p/q$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{P}$. Show that $q!(e - s_q)$ is an integer.

(iii) Obtain a contradiction using (1).

Problem 3. Let $\{a_n : n \in \mathbb{P}\}$ be an arbitrary collection of **non-negative** real numbers such that $\sum_{n=1}^{\infty} a_n$ converges. Determine which of the following series will necessarily converge (proof required), and which may either converge or diverge depending on the choice of the a_n's (examples required).

(a) $\sum_{n=1}^{\infty} a_n^2$

(b) $\sum_{n=1}^{\infty} \sqrt{a_n}$

(c) $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$

Problem 4. Show that each of the following series converges, and determine its sum.

(a) $\sum_{n=1}^{\infty} \frac{4n^2 - 1 + 3^{n-1}}{3^n\,(2n+1)\,(2n-1)}$

(b) $\sum_{n=6}^{\infty} \frac{6}{n^2 - 1}$

(c) $\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)(n+3)}$

Problem 5. For each of the series given below, determine whether it converges or diverges. You need not compute the sum in the case of convergence.

(1) $\sum_{n=1}^{\infty} \frac{n \sin^2(n\pi/3)}{2^n}$

(2) $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{1/n}$

(3) $\sum_{n=1}^{\infty} \frac{(-1)^n n^{25}}{(n+2)!}$

(4) $\sum_{n=5}^{\infty} \frac{\sqrt{n} + 1}{(n-1)(n+2)(n-4)}$

Note. You may use any of the tests stated in class. You may directly cite any examples discussed in class.