UM 101 HOMEWORK ASSIGNMENT 4
SKETCH OF SOLUTIONS

You might find the following inequalities useful. For fun(?), try to prove these on your own.

(i) Given $n \in \mathbb{N}$ and $x > 0$, $(1+x)^n \geq nx$.

(ii) Given $n \in \mathbb{N}$ and $x > 0$, $(1+x)^n \geq \frac{n(n-1)}{2}x^2$.

(iii) The AM-GM inequality. Given $x, y > 0$, $\frac{x+y}{2} \geq \sqrt{xy}$.

Problem 1. Let $\{a_n\}$ and $\{b_n\}$ be sequences in $\mathbb{R}$ such that for some $N \in \mathbb{N}$, $0 \leq a_n \leq b_n$ for all $n \geq N$. Convince yourself that if $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$. Using this fact, prove the following statements (you are not allowed to use logarithms for these proofs). Since $\lim_{n\to\infty} b_n = 0$, therefore for $\epsilon > 0$, there is an N_1 such that $|b_n| < \epsilon$ for all $n \geq N_1$. Now choose $M = max\{N_1, N\}$. So, for all $n \geq M$,

$$|a_n| = a_n \leq b_n = |b_n| < \epsilon \Rightarrow \lim_{n\to\infty} a_n = 0$$

(a) For any $r > 0$, $\lim_{n\to\infty} \sqrt[n]{r} = 1$. For $r > 1$, $a_n := \sqrt[n]{r} - 1 > 0$. So, by (i),

$$r = (1 + a_n)^n \geq na_n$$

Thus $0 \leq a_n \leq \frac{r}{n}$. Now apply the Squeeze Lemma stated above.

For $0 < r < 1$, $\frac{1}{r} > 1$. So, from above, $\lim_{n\to\infty} \sqrt[n]{\frac{1}{r}} = 1$. By using limit laws, we have $\lim_{n\to\infty} \sqrt[n]{r} = 1$.

(b) $\lim_{n\to\infty} \sqrt[n]{n} = 1$. Observe that $r_n := \sqrt[n]{n} - 1 > 0$. Thus, by (ii),

$$n = (1 + r_n)^n \geq \frac{n(n-1)}{2} r_n^2.$$

Thus, $0 \leq r_n \leq \frac{\sqrt{2}}{\sqrt{n-1}}$. Now apply the Squeeze Lemma stated above.

Problem 2. (a) Show that the series $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges. The mathematical constant e is defined as the sum of this series. Apply the Ratio Test.

(b) **Bonus (not for quiz).** Complete the following steps to show that $e = \sum_{n=0}^{\infty} \frac{1}{n!}$ is irrational, i.e., e cannot be written as p/q for any $p \in \mathbb{Z}$ and $q \in \mathbb{P}$.

(i) Let $s_n = 1 + \frac{1}{1!} + \cdots + \frac{1}{n!}$, $n \in \mathbb{N}$. Show that, for all $n \in \mathbb{P}$,

(1) $$0 < e - s_n < \frac{1}{n!n}.$$

(ii) Suppose $e = p/q$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{P}$. Show that $q!(e - s_q)$ is an integer.

(iii) Obtain a contradiction using (1).

See Rudin's *Principles of Mathematical Analysis*, III Ed., Theorem 3.32.

Problem 3. Let $\{a_n : n \in \mathbb{P}\}$ be an arbitrary collection of **non-negative** real numbers such that $\sum_{n=1}^{\infty} a_n$ converges. Determine which of the following series will necessarily converge (proof required), and which may either converge or diverge depending on the choice of the a_n's (examples required).

(a) $\sum_{n=1}^{\infty} a_n^2$

Method 1: Since $\sum_{n=1}^{\infty} a_n$ converges, therefore $\lim_{n\to\infty} a_n = 0$. So, for $\epsilon = 1$, there is an N_1 such that $|a_n| < 1$ for all $n \geq N_1$. Since all a_n's are non-negative real numbers, therefore

$$0 \leq a_n < 1 \text{ for all } n \geq N_1$$

Also $a_n = 0 \iff a_n^2 = 0$ and for $a_n \neq 0$, $a_n < 1 \Rightarrow a_n^2 < a_n$ for all $n \geq N_1$. So, we can say that for all $n \geq N_1$,

$$0 \leq a_n^2 \leq a_n$$

So, by comparison test, since $\sum_{n=1}^{\infty} a_n$ is convergent, $\sum_{n=1}^{\infty} a_n^2$ is also convergent.

Method 2: Consider $S_n = \sum_{k=1}^{n} a_n^2$ and $P_n = \sum_{k=1}^{n} a_n$ are the sequence of partial sums. So,

$\{P_n\}$ is convergent and hence $\{P_n^2\}$ is convergent, which gives that $\{P_n^2\}$ is bounded. So, there is an $M > 0$ such that $|P_n^2| = P_n^2 \leq M$ for all $n \in \mathbb{N}$.
Now since all a_n'n are non-negative, So, for all $n \in \mathbb{N}$,

$$S_n = \sum_{k=1}^{n} a_n^2 \leq (\sum_{k=1}^{n} a_n)^2 = P_n^2 \leq M$$

[You can prove this by mathematical induction and using $a^2 + b^2 \leq (a+b)^2$ and note that this is only true for non-negative numbers, otherwise find a counter example]

Which gives that $\{S_n\}$ is a bounded sequence. Also, we can observe that $\{S_n\}$ is monotonically increasing. Hence it is a convergent sequence. Therefore, the series $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) $\sum_{n=1}^{\infty} \sqrt{a_n}$

$\sum_{n=1}^{\infty} \sqrt{a_n}$ may or may not converge. Let $a_n = \frac{1}{n^2}$. Then $\sum a_n$ converges, but $\sum \sqrt{a_n}$ does not. For the second case, take $a_n = 1/n^4$. T

(c) $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$

For all $n \geq 1$, if we apply AM-GM inequality on a_n and $\frac{1}{n^2}$, we get,

$$0 \leq \sqrt{a_n \frac{1}{n^2}} \leq \frac{1}{2}(a_n + \frac{1}{n^2}) \Rightarrow 0 \leq \frac{\sqrt{a_n}}{n} \leq \frac{1}{2}(a_n + \frac{1}{n^2})$$

Since $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ are convergent, by the limit law of series, the sum $\sum_{n=1}^{\infty} (a_n + \frac{1}{n^2})$ is also convergent. Therefore, by the comparison test of series' (with $C = \frac{1}{2}$), $\frac{\sqrt{a_n}}{n}$ is convergent.

Problem 4. Show that each of the following series converges, and determine its sum.

(a) $\sum_{n=1}^{\infty} \frac{4n^2 - 1 + 3^{n-1}}{3^n\,(2n+1)\,(2n-1)}$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{4n^2 - 1 + 3^{n-1}}{3^n\,(2n+1)\,(2n-1)} &= \sum_{n=1}^{\infty} \frac{(2n+1)\,(2n-1) + 3^{n-1}}{3^n\,(2n+1)\,(2n-1)} \\
&= \sum_{n=1}^{\infty} \left\{ \frac{1}{3^n} + \frac{1}{3\,(2n+1)\,(2n-1)} \right\} \\
&\leq \sum_{n=1}^{\infty} \left\{ \frac{1}{3^n} + \frac{1}{3\,(n+1)\,(n)} \right\} \qquad (2n-1 \geq n \text{ and } 2n+1 \geq n+1 \;\forall\; n \geq 1) \\
&\leq \sum_{n=1}^{\infty} \left\{ \frac{1}{3^n} + \frac{1}{3\,n^2} \right\} \qquad (\frac{1}{n+1} \leq \frac{1}{n})
\end{aligned}$$

Now $\sum_{n=1}^{\infty} \frac{1}{3^n}$ is convergent because $\frac{1}{3} < 1$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by p-test. So, by limit laws, $\sum_{n=1}^{\infty} \left\{ \frac{1}{3^n} + \frac{1}{3\,n^2} \right\}$ is a convergent series. Therefore, by the comparison test,

$\sum_{n=1}^{\infty} \frac{4n^2 - 1 + 3^{n-1}}{3^n (2n+1)(2n-1)}$ is convergent.
Now to find it's sum, we will expand it as following;

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{4n^2 - 1 + 3^{n-1}}{3^n (2n+1)(2n-1)} &= \sum_{n=1}^{\infty} \frac{(2n+1)(2n-1) + 3^{n-1}}{3^n (2n+1)(2n-1)} \\ &= \sum_{n=1}^{\infty} \left\{ \frac{1}{3^n} + \frac{1}{3(2n+1)(2n-1)} \right\} \\ &= \sum_{n=1}^{\infty} \left\{ \frac{1}{3^n} + \frac{1}{6} \left\{ \frac{1}{(2n-1)} - \frac{1}{(2n+1)} \right\} \right\}\end{aligned}$$

We know that $\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{\frac{1}{3}}{1 - \frac{1}{3}} = \frac{1}{2}$ and let $S_n = \sum_{k=1}^{n} \frac{1}{6} \left\{ \frac{1}{(2k-1)} - \frac{1}{(2k+1)} \right\}$ be the sequence of partial sums of $\sum_{n=1}^{\infty} \left\{ \frac{1}{6} \left\{ \frac{1}{(2n-1)} - \frac{1}{(2n+1)} \right\} \right\}$.

So, we have;

$$S_n = \frac{1}{6} \left\{ (\frac{1}{1} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{5}) + \ldots + (\frac{1}{2n-3} - \frac{1}{2n-1}) + (\frac{1}{2n-1} - \frac{1}{2n+1}) \right\} = \frac{1}{6} \left\{ 1 - \frac{1}{2n+1} \right\}$$

Now, $\lim_{n \to +\infty} S_n = \frac{1}{6}(1-0) = \frac{1}{6}$. So, $\sum_{n=1}^{\infty} \left\{ \frac{1}{6} \left\{ \frac{1}{(2n-1)} - \frac{1}{(2n+1)} \right\} \right\}$ converges to $\frac{1}{6}$. Hence by using limit laws of series', we have;

$$\sum_{n=1}^{\infty} \frac{4n^2 - 1 + 3^{n-1}}{3^n (2n+1)(2n-1)} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$$

(b) $\sum_{n=6}^{\infty} \frac{6}{n^2 - 1}$ Do the same as part (a). The sum will be $\frac{11}{15}$.

(c) $\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)(n+3)}$ By partial fraction, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n}{(n+1)(n+2)(n+3)} &= \sum_{n=1}^{\infty} \left\{ \frac{-1}{2(n+1)} + \frac{2}{n+2} + \frac{-3}{2(n+3)} \right\} \\
&= \sum_{n=1}^{\infty} \left\{ \frac{-1}{2(n+1)} + \frac{\frac{1}{2}+\frac{3}{2}}{n+2} + \frac{-3}{2(n+3)} \right\} \\
&= \sum_{n=1}^{\infty} \left\{ (-\frac{1}{2})(\frac{1}{n+1} - \frac{1}{n+2}) + \frac{3}{2}(\frac{1}{n+2} - \frac{1}{n+3}) \right\}
\end{aligned}$$

Now $\sum_{n=1}^{\infty}(\frac{1}{n+1} - \frac{1}{n+2})$ and $\sum_{n=1}^{\infty}(\frac{1}{n+2} - \frac{1}{n+3})$ are both telescoping series' and hence convergent. So, by the limit laws of series', the given series is convergent. To find it's sum, do the same as (a). The sum will be $\frac{1}{4}$.

Problem 5. For each of the series given below, determine whether it converges or diverges. You need not compute the sum in the case of convergence.

(1) $\sum_{n=1}^{\infty} \frac{n \sin^2(n\pi/3)}{2^n}$

Note that

$$0 \leq \frac{n \sin^2(n\pi/3)}{2^n} \leq \frac{n}{2^n}.$$

Let $b_n = \frac{n}{2^n}$. Then,

$$\frac{b_{n+1}}{b_n} = \frac{(n+1)}{2n} = \frac{1+1/n}{2}.$$

By limit laws for sequences, the above sequence converges to $1/2 < 1$. By the ratio test, $\sum b_n < \infty$. Thus, by the comparison test, the given series **converges**.

(2) $\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^{1/n}$

Already shown in Problem 1 that $\lim_{n\to\infty} n^{1/n} = 1$. Thus, the individual terms in the above series are not converging to 0. The series **diverges**.

(3) $\sum_{n=1}^{\infty} \frac{(-1)^n n^{25}}{(n+2)!}$

We will show that the series **converges absolutely** by using the ratio test. Let $a_n = (-1)^n \frac{n^{25}}{(n+2)!}$. Then,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^{25}(n+2)!}{n^{25}(n+3)!} = \frac{(n+1)^{25}}{n^{26}+3n^{25}} = \frac{(1+1/n)^{25}}{3+1/n}.$$

By the limit laws for sequences, the above sequence converges to $1/3 < 1$. Thus, the ratio test, $\sum |a_n|$ converges. But, abs. cvg. => cvg.

(4) $\sum_{n=5}^{\infty} \frac{\sqrt{n}+1}{(n-1)(n+2)(n-4)}$

We use the comparison test, using the convergent series $\sum_{n=5}^{\infty} \frac{1}{n^{3/2}}$ for the comparison. Note that

$$\begin{aligned} 0 \le \frac{\sqrt{n}+1}{(n-1)(n+2)(n-4)} &= \frac{(n-1)}{(n-1)(n+2)(n-4)(\sqrt{n}-1)} && (n \neq 1) \\ &< \frac{1}{(n+2)(\sqrt{n}-1)} && (n-4 \geq 1, \text{ for } n \geq 5) \\ &< \frac{1}{n(\sqrt{n}-1)} && (n+2 > n) \\ &< \frac{2}{n^{3/2}} && (\sqrt{n}-1 > \tfrac{1}{2}\sqrt{n}, \text{ for } n \geq 5). \end{aligned}$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges, by limit laws $\sum_{n=1}^{\infty} \frac{2}{n^{3/2}}$ converges. Moreover, since dropping finitely many terms does not affect the convergence of the series, $\sum_{n=5}^{\infty} \frac{2}{n^{3/2}}$ converges. Thus, by the comparison test, the given series **converges**.