UM 101 HOMEWORK ASSIGNMENT 5

Posted on November 17, 2022
(NOT FOR SUBMISSION)

- These problems are for self-study.
- Some of these problems will be discussed at the next tutorial.
- A 15-minute quiz consisting of one problem from this assignment will be conducted at the end of the tutorial section.

You may freely use (without proof):

(*i*) $0 < \cos(x) < \left|\dfrac{\sin x}{x}\right| < 1$ for all $0 < |x| < \frac{\pi}{2}$.

(*ii*) Any trigonometric identities that you have seen in school.

(*iii*) Any limits computed in Lecture 12-14.

Problem 1 (Squeezing). Let f, g, h be functions defined on some neighborhood N of p, except perhaps at p. Suppose $f \leq g \leq h$ on N, and $\lim_{x\to p} f(x) = \lim_{x\to p} h(x) = a$ for some $a \in \mathbb{R}$. Show that

$$\lim_{x\to p} g(x) = a.$$

Remark. Feel free to use the above result (without proof) going forward.

Problem 2. In each of the following cases, determine whether the limit exists or not, and compute the limit whenever it exists. You may use any of the theorems stated in class, but state what you are using.

(*a*) $\displaystyle\lim_{x\to 2} \frac{(3x+1)^2 - 49}{x-2}$

(*b*) $\displaystyle\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$

(*c*) $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

(*d*) $\displaystyle\lim_{x\to p} x^n$ (for fixed $n \in \mathbb{N}$ and $p \in \mathbb{R}$)

Problem 3. Let f and g be functions on $\mathbb{R}$ such that

$$\lim_{x\to 0} f(x) = L \quad \text{and} \quad \lim_{y\to L} g(y) = M,$$

for some $L, M \in \mathbb{R}$. Is it true that

$$\lim_{x\to 0}(g \circ f)(x) = M?$$

If your answer is "yes", prove the above statement. If your answer is "no", provide a counterexample, and give a sufficient condition on g that will make the above statement true.

Problem 4 (Sequential characterization of continuity). Let $f : A \to \mathbb{R}$ be continuous at $p \in A$. Let $\{a_n\} \subset A$ be a sequence in A such that $\lim_{n\to\infty} a_n = p$. Show that $\lim_{n\to\infty} f(a_n) = f(p)$.
Remark. Feel free to use the above result going forward.

Problem 5. Complete the following steps to establish the continuity of the sine and cosine functions on $\mathbb{R}$. Recall (i) and (ii) given at the beginning of this assignment.

(a) Show that $\lim_{x\to 0} \sin x = 0$.

(b) Using (a) and a trignometric identity relating sin and cos, show that $\lim_{x\to 0} \cos x = 1$.

(c) Using (a) and (b), show that sin and cos are continuous at any $x \in \mathbb{R}$. (*Hint.* The statement $\lim_{x\to p} f(x) = L$ is equivalent to $\lim_{h\to 0} f(x+h) = L$.)

(d) Show that $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$.

Remark. Feel free to use the above results going forward.

Extra food for thought (not for the quiz)

Formulate and prove the converse of the statement in Problem 4. Be careful as there are multiple ways to interpret Problem 4 as an "if ..., then ..." statement.