UM 101 HOMEWORK ASSIGNMENT 5
SKETCH OF SOLUTIONS

Problem 1 (Squeezing). Let f, g, h be functions defined on some neighborhood N of p, except perhaps at p. Suppose $f \leq g \leq h$ on N, and $\lim_{x \to p} f(x) = \lim_{x \to p} h(x) = a$ for some $a \in \mathbb{R}$. Show that

$$\lim_{x \to p} g(x) = a.$$

Proof. Let $\epsilon > 0$ be given. As $\lim_{x \to p} f(x) = a$, there exists a $\delta_1 > 0$ such that $0 < |x - p| < \delta_1$ implies $|f(x) - a| < \epsilon$. i.e

$$a - \epsilon < f(x) < a + \epsilon \tag{1}$$

Similarly, we can say that there exists a $\delta_2 > 0$ such that $0 < |x - p| < \delta_2$ implies

$$a - \epsilon < h(x) < a + \epsilon \tag{2}$$

. Using (1), (2) and our hypothesis

$$a - \epsilon < f(x) \leq g(x) \leq h(x) < a + \epsilon$$

for $0 < |x - p| < \delta = \min\{\delta_1, \delta_2\}$. i.e $|g(x) - a| < \epsilon$. □

Problem 2. In each of the following cases, determine whether the limit exists or not, and compute the limit whenever it exists. You may use any of the theorems stated in class, but state what you are using.

(a) $\lim_{x \to 2} \frac{(3x+1)^2 - 49}{x - 2}$

Solution. **The limit is 42.** Observe that, for $x \neq 2$

$$\frac{(3x+1)^2 - 49}{x - 2} = \frac{9x^2 + 6x - 48}{x - 2} = \frac{(9x + 24)(x - 2)}{x - 2} = 9x + 24$$

. Now using algebra of limits, we have the result.

(b) $\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$

Solution. **The limit DNE.** Suppose the limit exists and is L. Let $\varepsilon = 1$. There exists a $\delta > 0$ such that whenever $0 < |x| < \delta$, we must have that $|\cos(1/x) - L| < 1$. By the Archimedean property of $\mathbb{R}$, there is an $N \in \mathbb{N}$ such that $1/n < \delta/\pi$ for all $n \geq N$. Thus,

$$2 = |\cos(N\pi) - \cos((N+1)\pi)| < 2.$$

This is absurd.

(c) $\lim_{x\to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

Solution. **The limit is 2.** Observe that, as long as $x \in (-1,1) \setminus \{0\}$,

$$\frac{\sqrt{1+x} - \sqrt{1-x}}{x} = \frac{2}{(\sqrt{1+x} + \sqrt{1-x})}.$$

Now $1+x$ & $1-x$ are polynomials, and thus admit limits everywhere on $\mathbb{R}$. When restricted to $(-1,1)$, their range lies in $[0,\infty)$. which is the domain of the continuous function $\sqrt{x}$. Thus, using that the composition of continuous functions is continuous, and the algebra of limits, we have the result.

(d) $\lim_{x\to p} x^n$ (for fixed $n \in \mathbb{N}$ and $p \in \mathbb{R}$)

Solution. **The limit is p^n.** We can either directly argue that since x^n is continuous, $\lim_{x\to p} x^n = p^n$, or we can take a more direct approach by observing that

$$|x^n - p^n| = |x-p| \left| \sum_{j=0}^{n-1} x^j p^{n-1-j} \right| \leq |x-p| \sum_{j=0}^{n-1} |x|^j |p|^{n-1-j}.$$

Trying giving an ε-δ proof using the above observation.

Problem 3. Let f and g be functions on $\mathbb{R}$ such that

$$\lim_{x\to 0} f(x) = L \quad \text{and} \quad \lim_{y\to L} g(y) = M,$$

for some $L, M \in \mathbb{R}$. Is it true that

$$\lim_{x\to 0} (g \circ f)(x) = M?$$

If your answer is "yes", prove the above statement. If your answer is "no", provide a counterexample, and give a sufficient condition on g that will make the above statement true.

No. Take $f(x) = 1$ for all $x \in \mathbb{R}$, and $g(x) = 2$, for $x \neq 1$, and $g(1) = 3$. Then, $L = 1$, $M = 2$, but $\lim_{x\to 0} g(f(x)) = \lim_{x\to 0} g(1) = 3$.

The above statement would be true if g were to be continuous at L. *Proof.* Let $\varepsilon > 0$. Since g is continuous at L and $g(L) = M$, there exist a $\tau > 0$ such that whenever $|y - L| < \tau$, we have that $|g(y) - g(L)| < \varepsilon$. Since $\lim_{x\to 0} f(x) = L$, there exists a $\delta > 0$ such the whenever $0 < |x| < \delta$, we have that $|f(x) - L| < \tau$. Combining the two statements yields the proof.

Problem 4 (Sequential characterization of continuity). Let $f : A \to \mathbb{R}$ be continuous at $p \in A$. Let $\{a_n\} \subset A$ be a sequence in A such that $\lim_{n\to\infty} a_n = p$. Show that $\lim_{n\to\infty} f(a_n) = f(p)$.

Proof. We need to show that for every $\epsilon > 0$, there exists an $n_0 \in N$ such that $|f(a_n) - f(p)| < \epsilon$ for every $n \geq n_0$.
Let $\epsilon > 0$ be given. Since f is continuous at p, there exists a $\delta > 0$, such that $|a - p| < \delta$ implies $|f(a) - f(p)| < \epsilon$. As $\lim_{n\to\infty} a_n = p$, there exists an $n_0 \in N$, such that $|a_n - p| < \delta$ for every $n \geq n_0$. Thus, $|f(a_n) - f(p)| < \epsilon$ for every $n \geq n_0$. □

Problem 5. Complete the following steps to establish the continuity of the sine and cosine functions on $\mathbb{R}$. Recall (i) and (ii) given at the beginning of this assignment.

(a) Show that $\lim_{x\to 0} \sin x = 0$. We are told that

$$0 < |\sin x| < |x|$$

for $|x| < \pi/2$. Let $\varepsilon > 0$. Choosing $\delta = \min\{\varepsilon, \pi/2\}$ does the trick.

(b) Using (a) and a trignometric identity relating sin and cos, show that $\lim_{x\to 0} \cos x = 1$. Hint: Use

$$\cos x = 1 - 2\sin^2(x/2).$$

(c) Using (a) and (b), show that sin and cos are continuous at any $x \in \mathbb{R}$. (*Hint.* The statement $\lim_{x\to p} f(x) = L$ is equivalent to $\lim_{h\to 0} f(p + h) = L$.) Hint: use that

$$|\sin(x+h) - \sin(x)| = 2\left|\cos\left(\frac{2x+h}{2}\right)\sin\left(\frac{h}{2}\right)\right| \leq 2\sin\left(\frac{h}{2}\right)$$

for $|h| < \pi/2$. A similar trick works for cos.

(d) Show that $\lim_{x\to 0} \frac{\sin x}{x} = 1$. We are told that

$$|\cos x| < \left|\frac{\sin x}{x}\right| < 1$$

for $0 < |x| < \pi/2$. But, $\sin(x) > 0$ when $0 < x < \pi/2$, and $\sin(x) < 0$ when $-\pi/2 < x < 0$ (since $\sin(x + \pi/2) = -\cos(x)$), we have that

$$|\cos(x)| < \frac{\sin x}{x} < 1$$

for $0 < |x| < \pi/2$. Now, use the fact that

$$||\cos x| - 1| \leq |\cos x - 1|$$

to say that $\lim_{x\to 0} |\cos x| = 1$, and complete the proof using the squeeze lemma.

Extra food for thought (not for the quiz)

Formulate and prove the converse of the statement in Problem 4. Be careful as there are multiple ways to interpret Problem 4 as an "if ..., then ..." statement.