UM 101 HOMEWORK ASSIGNMENT 6

Posted on November 24, 2022
(NOT FOR SUBMISSION)

- These problems are for self-study.
- Some of these problems will be discussed at the next tutorial.
- A 15-minute quiz consisting of one problem from this assignment will be conducted at the end of the tutorial section.

Problem 1. Give an example each of

(a) a bounded function $f : [-1, 1] \to \mathbb{R}$ that does not attain either its minimum or its maximum anywhere on $[-1, 1]$;

(b) a bounded continuous function $f : (-1, 1) \to \mathbb{R}$ that attains its minimum but does not attain its maximum on $(-1, 1)$.

Problem 2. Let f be a continuous function on $[a, b]$ such that $f(x) > 0$ for all $x \in [a, b]$. Show that there is a $c > 0$ such that $f(x) \geq c$ for all $x \in [a, b]$.

Problem 3. Show that every polynomial with real coefficients and odd degree has at least one real root.

Note. We haven't discussed the meaning of $\lim_{x\to\pm\infty} f(x)$, but you do know what it means to take the limits of the sequences $\{f(n)\}_{n\in\mathbb{N}}$ and $\{f(-n)\}_{n\in\mathbb{N}}$.

Problem 4. Let $f : [0, \pi/2] \to \mathbb{R}$ be given by

$$f(x) = \max\{x^2, \cos x\}.$$

Argue that f attains a global minimum on $[0, \pi/2]$ at some $c \in [0, \pi/2]$. Show that c is a solution of the equation $\cos x = x^2$.

Problem 5. For each given f below, determine its region of continuity and region of differentiability. As always, you may directly cite any theorems or examples discussed in class.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $h \circ g$, where $g(x) = x^3$ and $h(x) = |x|$.

(b) $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} |x|, & x < 0, \\ 0, & x = 0, \\ x^2 \cos\left(\frac{1}{x}\right), & x > 0. \end{cases}$$

(c) $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \dfrac{|\sin x|}{\sin |x|}, & x \neq n\pi, \text{ for any } n \in \mathbb{Z}, \\ 1, & \text{otherwise.} \end{cases}$$

Remark. We have not discussed one-sided limits/derivatives in class. While you may use those ideas, you do not need the language of one-sided limits to write your proofs.

Problem 6. Use induction to prove the following statement: given differentiable functions $f_1, ..., f_n$ on some interval (a, b), the function

$$g = \prod_{j=1}^{n} f_n = f_1 \cdot f_2 \cdot ... \cdot f_n$$

is also differentiable on (a, b), and

$$g' = \sum_{j=1}^{n} \left(f_j' \prod_{k=1, k\neq j}^{n} f_k \right).$$