UM 101 HOMEWORK ASSIGNMENT 6
SKETCH OF SOLUTIONS

Problem 1. Give an example each of

(a) a bounded function $f : [-1, 1] \to \mathbb{R}$ that does not attain either its minimum or its maximum anywhere on $[-1, 1]$;

Solution.

$$f(x) = \begin{cases} x, & x \in (-1, 1) \\ 0, & x = -1, 1. \end{cases}$$

Note that $\sup\{f(x) : x \in [-1, 1]\} = 1$ and $\inf\{f(x) : x \in [-1, 1]\} = -1$, yet f never attains these values.

(b) a bounded continuous function $f : (-1, 1) \to \mathbb{R}$ that attains its minimum but does not attain its maximum on $(-1, 1)$.

Solution.

$$f(x) = x^2.$$

Note that $f(x) \geq 0$ everywhere and $f(0) = 0$. Thus, f attains its minimum on $(-1, 1)$, but $\sup\{f(x) : x \in [-1, 1]\} = 1$, which is never attained on $(-1, 1)$.

Problem 2. Let f be a continuous function on $[a, b]$ such that $f(x) > 0$ for all $x \in [a, b]$. Show that there is a $c > 0$ such that $f(x) \geq c$ for all $x \in [a, b]$.

Proof. By algebra of limits $1/f$ is continuous on $[a, b]$. Since every continuous function on a closed and bounded interval is bounded, there exists an $M \in \mathbb{R}$ such that

$$0 < \frac{1}{f(x)} \leq M \qquad x \in [a, b].$$

Thus, $c = 1/M > 0$ and $f(x) \geq c$ for all $x \in [a, b]$. □

Problem 3. Show that every polynomial with real coefficients and odd degree has at least one real root.

Note. We haven't discussed the meaning of $\lim_{x \to \pm\infty} f(x)$, but you do know what it means to take the limits of the sequences $\{f(n)\}_{n \in \mathbb{N}}$ and $\{f(-n)\}_{n \in \mathbb{N}}$.

Proof. Let $f(x) = a_0 + a_1 x + \cdots + a_d x^d$, where d is odd, $a_0, ..., a_d \in \mathbb{R}$ and $a_d \neq 0$.
Case 1. $a_d > 0$. Note that

$$f(n) = a_d n^d \left(1 + \frac{a_{d-1}}{a_d n} + \cdots + \frac{a_0}{n^d}\right).$$

By algebra of sequential limits, and the fact that $\lim_{n\to\infty} n^{-p} = 0$ for $p > 0$, we have the existence of some $N_1 \in \mathbb{P}$ such that

$$1 + \frac{1}{2} > 1 + \frac{a_{d-1}}{a_d n} + \cdots + \frac{a_0}{n^d} > 1 - \frac{1}{2}$$

for all $n \geq N_1$. Thus,

$$f(N_1) > \frac{a_d}{2} N^d > 0.$$

By the same reasoning, there is an $N_2 \in \mathbb{P}$ such that

$$f(-N_2) < \frac{3}{2} a_d (-N_2)^d < 0$$

since d is odd. Since f is continuous (it's a polynomial) on $[-N_2, N_1]$ and it takes opposite signs on the endpoints, by IVT, it must vanish somewhere in $(-N_2, N_1)$. □

Problem 4. Let $f : [0, \pi/2] \to \mathbb{R}$ be given by

$$f(x) = \max\{x^2, \cos x\}.$$

Argue that f attains a global minimum on $[0, \pi/2]$ at some $c \in [0, \pi/2]$. Show that c is a solution of the equation $\cos x = x^2$.

We will use the fact that

$$g(x) = |x| = \begin{cases} x, \ x > 0, \\ -x, \ x \leq 0 \end{cases}$$

is continuous. This is so because g is a polynomial on $(-\infty, 0) \cup (0, \infty)$, and at 0, we use that for every $\varepsilon > 0$, if we choose $\delta = \varepsilon$, then $|x| < \delta \Rightarrow |g(x)| = |x| < \varepsilon$.
Now, observe that

$$f(x) = \frac{x^2 + \cos x + |x^2 - \cos x|}{2}.$$

Thus, by the algebra of continuous functions, f is continuous. Since every continuous function on a closed and bounded interval attains its minimum and maximum, the claimed c exists.

Now say that $c^2 \neq \cos c$. **Case 1.** $c^2 > \cos c$**.** Since $\cos x \geq 0$ for $x \in [0, \pi/2]$, $c \neq 0$. By the continuity of $h(x) = x^2 - \cos x$, corresponding to $\varepsilon = h(c)/2$, there is a $\delta > 0$ such that

$$0 < \frac{h(c)}{2} < h(x)$$

for all $c-\delta < x \le c$. Thus, $f(x) = x^2$ for $c-\delta < x \le c$, but $(c-\delta)^2 < c^2$, and c is a global minimum of f!

Case 2. $c^2 < \cos c$. Since $(\pi)^2/4 > \cos(\pi/2)$, so $c \neq \pi/2$. By a similar argument as in Case 1, we get a $0 < \delta < \pi/2$ such that $f(x) = \cos x$ for $c \le x < c+\delta$. But,

$$\cos(c+\delta) = \cos(c)\cos(\delta) - \sin(c)\sin(\delta) < \cos(c).$$

But f attains its global min. at c!

Problem 5. For each given f below, determine its region of continuity and region of differentiability. As always, you may directly cite any theorems or examples discussed in class.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined by $h \circ g$, where $g(x) = x^3$ and $h(x) = |x|$.

Claim. f is continuous and differentiable on $\mathbb{R}$.

Since g is a polynomial, h has been shown to be continuous in Problem 5., and the composition of continuous functions is continuous, $f = h \circ g$ is continuous.

Note that

$$f(x) = \begin{cases} x^3, & x > 0, \\ -x^3, & x \le 0. \end{cases}$$

Since polynomials are differentiable on open intervals, we only need to check the differentiability of f at 0. Let $\varepsilon > 0$. Choose $\delta = \sqrt{\varepsilon} > 0$. Then, whenever $|k| < \delta$, we have that

$$\left|\frac{f(k)}{k}\right| = k^2 < \varepsilon.$$

Thus, f is also diff. at 0.

(b) $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} |x|, & x < 0, \\ 0, & x = 0, \\ x^2 \cos\left(\frac{1}{x}\right), & x > 0. \end{cases}$$

Claim. f is continuous and differentiable on $\mathbb{R} \setminus \{0\}$.

The continuity of f on $\mathbb{R} \setminus \{0\}$ follows from known results. We check the continuity of f at 0. Let $\varepsilon > 0$. Choose $\delta = \min\{\varepsilon, \sqrt{\varepsilon}\}$. Then, whenever $|x| < \delta$, either

(i) $0 \le x < \delta$, in which case $f(x) = |x| = x < \varepsilon$, or

(ii) $-\delta < x < 0$, in which case $|f(x)| \le |x|^2 < \delta^2 < \varepsilon$ (since $\cos(x) \le 1$ for all x).

Next, we determine the region of differentiability of f. For $x, y > 0$ and $x \neq y$,

$$\begin{aligned}\frac{\cos(1/x) - \cos(1/y)}{x-y} &= \frac{2\sin((x+y)/xy)\sin((x-y)/xy)}{x-y} \\ &= \frac{2xy\sin((x+y)/xy)\sin((x-y)/xy)}{xy(x-y)}.\end{aligned}$$

Now by the algebra of limits, and the fact that $\lim_{x\to 0}\sin(x)/x = 1$, we have that

$$\lim_{y\to x}\frac{\cos(1/x)-\cos(1/y)}{x-y}$$

exists, and $\cos(1/x)$ is differentiable at each $x > 0$. By the algebra of differentiable functions, we have that f is differentiable on $\mathbb{R}\setminus\{0\}$.

Let $h \neq 0$. Then

$$g(h) = \frac{f(h)}{h} = \begin{cases} -1, \ h < 0, \\ h\cos(1/h), \ h > 0. \end{cases}$$

Consider the sequence $\left\{a_n = (-1)^n \frac{2}{(2n+1)\pi}\right\}$. Then, $\{g(a_n)\}$ is the oscilating sequence $\{-1, 0, -1, 0, ...\}$ which does not converge. Thus, by the seq. char. of limits, f is not diff. at 0.

(c) $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} \dfrac{|\sin x|}{\sin|x|}, & x \neq n\pi, \text{ for any } n \in \mathbb{Z}, \\ 1, & \text{otherwise.} \end{cases}$$

From trig. identities, we know that $\sin(x) > 0$ on $x \in (2k\pi, (2k+1)\pi)$ and $\sin(x) < 0$ for $x \in ((2k+1)\pi, (2k+2)\pi)$ for each $k \in \mathbb{Z}$. Thus,

$$f(x) = \begin{cases} +1, \ x \in [-(2k+1)\pi, 2k\pi] \cup [2k\pi, (2k+1)\pi], k \in \mathbb{N}, \\ -1, \ x \in (-(2k+2)\pi, -(2k+1)\pi) \cup ((2k+1)\pi, (2k+2)\pi), k \in \mathbb{N}. \end{cases}$$

Using similar techniques as in the previous problems, you can show that f is continuous and differentiable precisely on $\mathbb{R}\setminus\{n\pi : n \in \mathbb{Z}\setminus\{0\}\}$.

Problem 6. Use induction to prove the following statement: given differentiable functions $f_1, ..., f_n$ on some interval (a, b), the function

$$g = \prod_{j=1}^{n} f_n = f_1 \cdot f_2 \cdot ... \cdot f_n$$

is also differentiable on (a, b), and

$$g' = \sum_{j=1}^{n} \left(f_j' \prod_{k=1, k \neq j}^{n} f_k \right).$$

This is a routine application of indution.