UM 101 HOMEWORK ASSIGNMENT 7

Posted on December 15, 2022
(NOT FOR SUBMISSION)

- These problems are for self-study.
- Some of these problems will be discussed at the next tutorial.
- A 15-minute quiz consisting of one problem from this assignment will be conducted at the end of the tutorial section.

Problem 1. Let $f : (a, b) \to \mathbb{R}$ be a function and $c \in (a, b)$. Which of the following statements are true, and which are false.

(a) If $\lim_{h\to 0} \frac{f(c) - f(c-h)}{h}$ exists, then f is differentiable at c and $f'(c) = \lim_{h\to 0} \frac{f(c) - f(c-h)}{h}$.

(b) If $\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} n\,(f(c + 1/n) - f(c))$ exists, then f is differentiable at c.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. In each of the following cases, argue that g is differentiable on its domain (you may use any theorems stated in class), and determine the derivative of g in terms of f'.

(a) $g(x) = f(x^3) + \sin(f(x))$.
(b) $g(x) = (f \circ f)(x)$.

Problem 3. Show that $f(x) = x^{1/3}$, $x \in \mathbb{R}$, is not differentiable at $x = 0$.

Problem 4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at c and $f(c) = 0$. Show that $g(x) = |f(x)|$ is differentiable at c if and only if $f'(c) = 0$.

Problem 5. Let $a > b > 0$ and $n \in \mathbb{N}$, $n \geq 2$. Show that

$$a^{\frac{1}{n}} - b^{\frac{1}{n}} < (a - b)^{\frac{1}{n}}.$$

Hint. Consider the function $x^{\frac{1}{n}} - (x-1)^{\frac{1}{n}}$ on the interval $[1, a/b]$.

Problem 6. Let $a_1 < a_2 < \cdots < a_n$ be real numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \sum_{j=1}^{n} |a_j - x|.$$

Find the point(s) of global minimum of f.

Hint. Draw the graphs of some examples. Writing f as a piecewise function will help.

Definition. Given a function $(a, b) \to \mathbb{R}$ and $c \in (a, b)$ we say that

$$\lim_{h \to c} f(h) = +\infty$$

if, for every $M > 0$, there is a $\delta_M > 0$ such that $f(x) > M$ for all $x \in N_{\delta_M}(c) \cap (a, b)$.

Problem 7. Let $f : [a, b] \to \mathbb{R}$ be a continuous and invertible function. Let $p \in (a, b)$. The following two statements were proposed in class. Prove both of them.

(a) If

$$\lim_{h \to 0} \frac{f(p+h) - f(p)}{h} = +\infty,$$

then f^{-1} is differentiable at $q = f(p)$ and $\left(f^{-1}\right)'(q) = 0$.

(b) If f is differentiable at p and $f'(p) = 0$, then f^{-1} is not differentiable at $q = f(p)$.