UM 101 HOMEWORK ASSIGNMENT 7
SKETCH OF SOLUTIONS

Problem 1. Let $f : (a, b) \to \mathbb{R}$ be a function and $c \in (a, b)$. Which of the following statements are true, and which are false.

(a) If $\lim_{h\to 0} \frac{f(c) - f(c-h)}{h}$ exists, then f is differentiable at c and $f'(c) = \lim_{h\to 0} \frac{f(c) - f(c-h)}{h}$.

Solution: Consider the function $g : \mathbb{R} \longrightarrow \mathbb{R}$ defined as $g(x) = -x$ for every $x \in \mathbb{R}$. Note that $\lim_{h\to 0} g(h) = 0$. For the given c, define the function $g_0(x) = \frac{f(c) - f(c-x)}{x}$ for $x \neq 0$. Now use the following composition theorem:
Let f and g be functions such that

$$\lim_{x\to p} f(x) = L \quad \text{and} \quad \lim_{y\to L} g(y) = M.$$

Moreover, suppose that for some $\delta > 0$, if $0 < |x - p| < \delta$, then $|f(x) - L| > 0$. Then,

$$\lim_{x\to p} g(f(x)) = \lim_{y\to L} g(y).$$

(b) If $\lim_{\substack{n\to\infty \\ n\in\mathbb{N}}} n\,(f(c + 1/n) - f(c))$ exists, then f is differentiable at c.

Solution: The given limit exists holds for the function $f(x) = |x|$ at $c = 0$, but $f(x) = |x|$ is not differentiable at 0.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. In each of the following cases, argue that g is differentiable on its domain (you may use any theorems stated in class), and determine the derivative of g in terms of f'.

(a) $g(x) = f(x^3) + \sin(f(x))$.

Solution: You need to check the following before applying chain rule of differentiation to the composition $f \circ h$ of functions f and h:

- The range of h is a subset of the domain of f.
- While checking the differentiability of $f \circ h$ at a point c, you need to check if h is differentiable at c and if f is differentiable at $h(c)$.

Check the above and use chain rule and algebra of differentiable functions to see that $g'(x) = f'(x^3)3x^2 + \cos(f(x))f'(x)$ for every $x \in \mathbb{R}$

(b) $g(x) = (f \circ f)(x)$.

Solution: Easy.

Problem 3. Show that $f(x) = x^{1/3}$, $x \in \mathbb{R}$, is not differentiable at $x = 0$.

Solution: We need to show that $f(x) = x^{1/3}$, $x \in \mathbb{R}$, is not differential at $x = 0$. Suppose not, , i.e., it is differentiable, or the limit $\lim_{h\to 0} \dfrac{f(0+h) - f(0)}{h}$ exists, and the limit is denoted by $f'(0)$. Thus, for every $\epsilon > 0$, there exists a $\delta > 0$ such that for any h satisfying $0 < |h| < \delta$, $\left|\dfrac{f(0+h) - f(0)}{h} - f'(0)\right| < \epsilon$ holds, i.e $\left|\dfrac{h^{1/3}}{h} - f'(0)\right| < \epsilon$, i.e. $\left|\dfrac{1}{h^{2/3}} - f'(0)\right| < \epsilon$. Now proceed as it was shown is class as how for $f(x) = \frac{1}{x}$, $x \neq 0$, $\lim_{x\to 0} f(x)$ does not exist.

Problem 4. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable at c and $f(c) = 0$. Show that $g(x) = |f(x)|$ is differentiable at c if and only if $f'(c) = 0$.

Solution: Suppose g is differentiable at c. We claim that $g'(c) = 0$. Suppose not, i.e., suppose $g'(c) = L \neq 0$. Then, either $L > 0$ or $L < 0$. In the case that $L > 0$, choose $\epsilon > 0$ such that $L - \epsilon > 0$. There is a $\delta > 0$ such that whenever $0 < h < \delta$, we have that

$$\left|\tfrac{g(c+h)-g(c)}{h} - L\right| = \left|\tfrac{g(c+h)}{h} - L\right| < \epsilon$$

$$\text{or } 0 < L - \epsilon < \tfrac{g(c+h)}{h} < L + \epsilon$$

$$\text{or } 0 < L - \epsilon < \tfrac{|f(c+h)|}{h} < L + \epsilon$$

But $\frac{|f(c+h)|}{h} \leq 0$ for $-\delta < h < 0$, which contradicts the above statement.

The case $L < 0$ can be dealt with similarly.

Now, we have a $\delta > 0$ such that whenever $0 < h < \delta$ then

$$\left|\frac{g(c+h)}{h}\right| < \epsilon \Rightarrow \left|\frac{|f(c+h)|}{h}\right| < \epsilon \Rightarrow \frac{\Big||f(c+h)|\Big|}{|h|} < \epsilon \Rightarrow \frac{|f(c+h) - f(c)|}{|h|} < \epsilon \Rightarrow \left|\frac{f(c+h) - f(c)}{h}\right| < \epsilon$$

Therefore $\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = 0 \Rightarrow f'(c) = 0$
Conversely, when $f'(c) = 0$ then for a given $\epsilon > 0$, there is a $\delta > 0$ such that

$$0 < |h| < \delta \Rightarrow \left|\frac{f(c+h) - f(c)}{h}\right| < \epsilon \Rightarrow \left|\frac{f(c+h)}{h}\right| < \epsilon \Rightarrow \frac{|f(c+h)|}{|h|} < \epsilon \quad \Rightarrow \frac{\left||f(c+h)|\right|}{|h|} < \epsilon$$
$$\Rightarrow \left|\frac{|f(c+h)|}{h}\right| < \epsilon$$
$$\Rightarrow \left|\frac{g(c+h)-g(c)}{h}\right| < \epsilon$$

Hence, g is differentiable at c and $g'(c) = 0$.

Problem 5. Let $a > b > 0$ and $n \in \mathbb{N}$, $n \geq 2$. Show that

$$a^{\frac{1}{n}} - b^{\frac{1}{n}} < (a-b)^{\frac{1}{n}}.$$

Hint. Consider the function $x^{\frac{1}{n}} - (x-1)^{\frac{1}{n}}$ on the interval $[1, a/b]$.

Solution: Since it is done in the class that for $x > 0$, $x^{\frac{1}{n}}$ is differentiable and $f'(x) = \frac{1}{nx^{\frac{n-1}{n}}}$. So, both $x^{\frac{1}{n}}$ and $(x-1)^{\frac{1}{n}}$ are differentiable in $(1, \frac{a}{b})$, then by algebra of derivatives, the function $h(x) = x^{\frac{1}{n}} - (x-1)^{\frac{1}{n}}$ is differentiable in $(1, a/b)$ and is continuous in $[1, a/b]$ (Prove that $(x-1)^{\frac{1}{n}}$ is continuous at $x = 1$).
Apply Mean Value Theorem and you will get the required result.

Problem 6. Let $a_1 < a_2 < \cdots < a_n$ be real numbers. Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \sum_{j=1}^{n} |a_j - x|.$$

Find the point(s) of global minimum of f.

Hint. Draw the graphs of some examples. Writing f as a piecewise function will help.
Solution: First we will write f as a piecewise function. We have two cases.

Case 1. n is even, i.e $n = 2k$ for some $k \in \mathbb{N}$.

$$f(x) = \begin{cases} a_1 + a_2 + a_3 + \cdots + a_n - nx, \; x \leq a_1, \\ -a_1 + a_2 + a_3 + \cdots + a_n - (n - 2 \times 1)x, \; x \in [a_1, a_2], \\ -a_1 - a_2 + a_3 + a_4 \cdots + a_n - (n - 2 \times 2)x, \; x \in [a_2, a_3] \\ \vdots \\ -a_1 - a_2 - a_3 \cdots - a_k - a_{k+1} + a_{k+2} + \cdots + a_n - (n - 2 \times (k-1))x, \; x \in [a_{k-1}, a_k], \\ -a_1 - a_2 - a_3 \cdots - a_k + a_{k+1} + \cdots + a_n, \; x \in [a_k, a_{k+1}], \\ -a_1 - a_2 - a_3 \cdots - a_k - a_{k+1} + a_{k+2} + \cdots + a_n - (n - 2 \times (k+1))x, \; x \in [a_{k+1}, a_{k+2}] \\ \vdots \\ nx - a_1 - a_2 - \cdots - a_n, \; x \geq a_n \end{cases}$$

Answer. All $x \in [a_k, a_{k+1}]$.

- Since f is a continuous function on $[a_1, a_n]$, there is a $c \in [a_1, a_2]$ such that

$$f(x) \geq f(c) \; \forall x \in [a_1, a_n].$$

- Since $f'(x) = -n < 0$ on $(-\infty, a_1)$ and $f'(x) = n > 0$ on (a_n, ∞), we have that

$$f(x) \geq f(a_1) \geq f(x) \; \forall x < a_1,$$
$$f(x) \geq f(a_n) \geq f(c) \; \forall x > a_n.$$

  Thus, the points of global minimum of f on $\mathbb{R}$ are the points of global minimum of f on $[a_1, a_n]$.
- The potential points of local extrema are: $a_1, a_2, ..., a_n$ (since they are points of non-differentiability) and all the points in (a_k, a_{k+1}) since f' vanishes there.
- Now study the sign of f' to conclude that f is strictly decreasing on $(-infty, a_k)$ and strictly increasing on (a_{k+1}, ∞).

Case 2. n is odd, i.e $n = 2k+1$ for some $k \in \mathbb{N}$.

$$f(x) = \begin{cases} a_1 + a_2 + a_3 + \cdots + a_n - nx, \ x \le a_1, \\ -a_1 + a_2 + a_3 + \cdots + a_n - (n - 2\times 1)x, \ x \in [a_1, a_2], \\ -a_1 - a_2 + a_3 + a_4 \cdots + a_n - (n - 2.2)x, \ x \in [a_2, a_3] \\ \vdots \\ -a_1 - a_2 - a_3 \cdots - a_k + a_{k+1} + \cdots + a_n - x, x \in [a_k, a_{k+1}], \\ -a_1 - a_2 \cdots - a_k + a_{k+2} + a_{k+3} \cdots + a_n, \ x = a_{k+1} \\ -a_1 - a_2 \cdots - a_k - a_{k+1} + a_{k+2} + a_{k+3} \cdots + a_n + x, x \in [a_{k+1}, a_{k+2}] \\ \vdots \\ nx - a_1 - a_2 - \cdots - a_n, \ x_n \end{cases}$$

Answer. $x = a_{k+1}$.
Repeat the same technique.

Definition. Given a function $(a,b) \to \mathbb{R}$ and $c \in (a,b)$ we say that

$$\lim_{h\to c} f(h) = +\infty$$

if, for every $M > 0$, there is a $\delta_M > 0$ such that $f(x) > M$ for all $x \in N_{\delta_M}(c) \cap (a,b)$.

Problem 7. Let $f : [a,b] \to \mathbb{R}$ be a continuous and invertible function. Let $p \in (a,b)$. The following two statements were proposed in class. Prove both of them.

(a) If

$$\lim_{h\to 0} \frac{f(p+h) - f(p)}{h} = +\infty,$$

then f^{-1} is differentiable at $q = f(p)$ and $(f^{-1})'(q) = 0$.

Solution: For a given $\epsilon > 0$, let $M = \frac{1}{\epsilon} > 0$. Then, there is a δ_ϵ such that for $h \in N_{\delta_\epsilon} \cap [a,b]$, $\frac{f(p+h)-f(p)}{h} > \frac{1}{\epsilon}$.
By the inverse function theorem, $f^{-1} : J \to [a,b]$ is continuous and one-one, onto. Let $h(k) = f^{-1}(q+k) - f^{-1}(q)$. Then, $h(k) \neq 0$ whenever $k \neq 0$, (since f^{-1} is is one-to-one). Then $h(k) + p = f^{-1}(q+k)$, or $k = f(h(k)+p) - f(p)$. Thus,

$$\frac{f^{-1}(q+k) - f^{-1}(q)}{k} = \frac{h(k)}{f(p+h(k)) - f(p)} = \frac{1}{\frac{f(p+h(k))-f(p)}{h(k)}}.$$

Since f^{-1} is continuous, so, $h(k)$ is continuous. So, for δ_ϵ, there is a $\delta_1 > 0$ such that whenever $|k| < \delta_1$, $|h(k) - h(0)| = |h(k)| < \delta_\epsilon \Rightarrow \frac{f(p+h(k))-f(p)}{h(k)} > \frac{1}{\epsilon} > 0$.

Therefore, whenever $0 < |k| < \delta_1$ then, $\left|\frac{f^{-1}(q+k)-f^{-1}(q)}{k}\right| = \left|\frac{1}{\frac{f(p+h(k))-f(p)}{h(k)}}\right| < \epsilon$. We are done.

(b) If f is differentiable at p and $f'(p) = 0$, then f^{-1} is not differentiable at $q = f(p)$.

Solution: Let $M > 0$ and choose $\epsilon = \frac{1}{M} > 0$. Then, there is a $\delta > 0$ such that

$$0 < |h| < \delta \;\Rightarrow\; \left|\frac{f(p+h)-f(p)}{h}\right| < \epsilon$$

Take $h(k)$ as in part (a). Then, by the continuity of $h(k)$, there is a $\delta_M > 0$ such that

$$|k| < \delta_M \;\Rightarrow\; |h(k) - h(0)| = |h(k)| < \delta.$$

This gives $\left|\frac{f(p+h(k))-f(p)}{h(k)}\right| < \epsilon \Rightarrow -\epsilon < \frac{f(p+h(k))-f(p)}{h(k)} < \epsilon$.

Therefore,

$$|k| < \delta_M \;\Rightarrow\; \frac{f^{-1}(q+k)-f^{-1}(q)}{k} = \frac{1}{\frac{f(p+h(k))-f(p)}{h(k)}} > \frac{1}{\epsilon} = M$$

So,

$$\lim_{k\to 0} \frac{f^{-1}(q+k)-f^{-1}(q)}{k} = +\infty,$$

Hence f^{-1} is not differentiable at $q = f(p)$.

Note. We are using that $\lim_{x\to c} f(x) = +\infty \Rightarrow$ the limit of f as x approaches c does not exist! Prove this using sequential characterization of limits (or any other method).