UM 101 HOMEWORK ASSIGNMENT 8

Posted on December 23, 2022
(NOT FOR SUBMISSION)

- These problems are for self-study. Some of these problems are long, so you may need more than a week to work on this assignment.
- There will be no quiz on this material.

Problem 1. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, c) \cup (c, b)$. Show that if $\lim_{x\to c} f'(x) = L$, then $f'(c)$ exists and equals L.

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2x^4 + x^4 \sin\left(\frac{1}{x}\right), \ x \neq 0, \\ 0, \ x = 0. \end{cases}$$

Show that f has a global minumum at $x = 0$, but for every $\delta > 0$, f is not monotone on either $(-\delta, 0)$ or on $(0, \delta)$. **Note.** This example demonstrates the limitations of the first derivative test.

Problem 3. Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Show that for any $x, y \in (a, b)$, if k is a number between $f'(x)$ and $f'(y)$, then there is some $c \in (x, y)$ such that $f'(c) = k$. **Note.** This means the a derivative function has the intermediate value property. This should help you construct a function that is not continuous but has the intermediate value property!

Hint. See Apostol, Section 4.15, Exercise 10.

Problem 4. Show that

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1 + x} \leq 1 + \frac{1}{2}x$$

for all $x > 0$.

Problem 5. Locate and classify all the points of local extrema of the function

$$f(x) = x|x^2 - 12|$$

on the domain $[-2, 3]$.

$*****$The following problems may require more time than the rest.$*****$

Problem 6 (Proof of Taylor's Theorem). Recall the following theorem. Let $f : (a, b) \to \mathbb{R}$ be $(n+1)$-times differentiable on (a, b). Let $x_0 \in (a, b)$. Then, for any $x \in (a, b)$, there exists a c_x between x and x_0 such that

$$f(x) = P_n^{x_0}(x) + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0)^{n+1}.$$

Brainstorming. WLOG assume $x < x_0$. Set

$$K_n = (n+1)!\frac{f(x) - P_n^{x_0}(x)}{(x - x_0)^{n+1}}.$$

We will try to apply MVT to an appropriate function $G : [x, x_0] \to \mathbb{R}$ that satisfies two conditions:

(i) $G(x) = G(x_0)$

(ii) $G'(c) = 0$ implies that

$$f^{(n+1)}(c) = K_n. \tag{1}$$

In class, we observed that the naive idea of $G(t) = f^{(n)}(t) - tK_n$ does not satisfy (i).

Let us attempt a new G for the case $n = 1$. Remember that x is fixed and the variable of differentiation is t in the below argument. Note that we can write (**??**) as

$$\begin{aligned}
& f''(c) = K_1 \\
\iff\ & (x-c)f''(c) = (x-c)K_1 \\
\iff\ & \Big[f(t) + (x-t)f'(t)\Big]'_{t=c} = -K_1\Big[\frac{(x-t)^2}{2}\Big]'_{t=c} \\
\iff\ & \Big[P_1^t(x) + (x-t)^2\frac{K_1}{2!}\Big]'_{t=c} = 0
\end{aligned}$$

(a) Use the function $G(t) = P_1^t(x) + \frac{(x-t)^2 K_2}{2!}$ to prove Taylor's theorem in the case of $n = 1$.

(b) By modifying G suitably for each n, prove Taylor's theorem. *Hint.* It may help to try $n = 2$ before attempting the general case.

Problem 7. Let $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. Consider the following "definitions". In each case, determine whether it is equivalent to the definition given in class. If yes, provide a proof. If not, provide an example.

(i) We say that f admits a limit as x approaches c if for every $\varepsilon > 0$, there exists an $L \in \mathbb{R}$ and a $\delta > 0$ such that for every $x \in N_\delta(c) \setminus \{c\}$, we have that

$$|f(x) - L| < \varepsilon.$$

(*ii*) We say that f admits a limit as x approaches c if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $x \in N_\delta(c) \setminus \{c\}$ there exists an $L \in \mathbb{R}$ such that

$$|f(x) - L| < \varepsilon.$$

(*iii*) We say that f admits a limit as x approaches c if there exists an $L > 0$ and a $\delta > 0$, such that for every $\varepsilon > 0$, whenever $x \in N_\delta(c) \setminus \{c\}$, we have that

$$|f(x) - L| < \varepsilon.$$

Note. The text in red marks the main departure from the original definition.