UM 101 HOMEWORK ASSIGNMENT 8
SKETCH OF SOLUTIONS

Problem 1. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on $(a, c) \cup (c, b)$. Show that if $\lim_{x \to c} f'(x) = L$, then $f'(c)$ exists and equals L.

Proof. Let $g : (a, c) \cup (c, b) \to \mathbb{R}$ be given by

$$g(x) = \frac{f(x) - f(c)}{x - c}.$$

We want to show that $\lim_{x \to c} g(c) = L$.
Given $x \in (a, c)$, MVT applied to $[x, c]$ gives a $p(x) \in (x, c)$ such that

$$f'(p(x)) = \frac{f(x) - f(c)}{x - c} = g(x).$$

Similarly, for each $x \in (c, b)$, there is a $p(x) \in (c, x)$ such that $f'(p(x)) = g(x)$. Thus, we have a function $p : (a, c) \cup (c, b) \to \mathbb{R}$ such that

$$f'(p(x)) = g(x).$$

Observe the following features of p:

(*a*) $p(x) \neq c$ for every $x \in (a, c) \cup (c, b)$.
(*b*) $0 \leq |p(x) - c| \leq |x - c|$ for all $x \in (a, c) \cup (c, b)$. Thus, by the squeeze lemma for functions and the continuity of $|\cdot|$ and polynomials, we have that $\lim_{x \to c} p(x) = c$.

By the composition rule for limits (where the "inside" function satisfies the relevant condition: see item (*a*) above), we have that

$$\lim_{x \to c} g(x) = \lim_{x \to c} f'(p(x)) = \lim_{y \to c} f'(y) = L.$$

□

Problem 2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 2x^4 + x^4 \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Show that f has a global minumum at $x = 0$, but for every $\delta > 0$, f is not monotone on either $(-\delta, 0)$ or on $(0, \delta)$. **Note.** This example demonstrates the limitations of the first derivative test.

Proof. Since $-1 \leq \sin(y) \leq 1$ for all $y \in \mathbb{R}$, we have that for $x \neq 0$,

$$f(x) = 2x^4 + x^4 \sin(1/x) \geq 2x^4 - x^4 \geq 0 = f(0).$$

Thus, $x = 0$ is point of global minimum.
Suppose there exists a $\delta > 0$ such that f is monotone on either $(-\delta, 0)$ or $(0, \delta)$. Then, since f is a composition of differentiable functions on $\mathbb{R} \setminus \{0\}$, f' would take the same sign on either $(-\delta, 0)$ or $(\delta, 0)$ (by the characterization of monotoncity in terms of first derivatives). Note that

$$f'(x) = 8x^3 + 4x^3 \sin\left(\frac{1}{x}\right) - x^2 \cos\left(\frac{1}{x}\right), \; x \neq 0.$$

Thus, $f'\left(\frac{2}{(4k+1)\pi}\right) = 12\left(\frac{2}{(4k+1)\pi}\right)^3 > 0$ for all $k \in \mathbb{Z}$, and $f'\left(\frac{1}{2k\pi}\right) = 8\left(\frac{1}{2k\pi}\right)^3 - \left(\frac{1}{2k\pi}\right)^2 < 0$ for all integers k satisfying $|k| \geq 2$. Now choose an integer $K \geq 2$ such that

$$\min\left\{x_K = \frac{2}{(4K+1)\pi}, y_K = \frac{1}{2K\pi}\right\} < \delta.$$

Then, $x_K, y_K \in (0, \delta)$, but $f'(x_K)$ and $f'(y_K)$ take opposite signs. Similarly, $-x_K, -y_K \in (-\delta, 0)$, but but $f'(-x_K)$ and $f'(-y_K)$ take opposite signs. This contradicts our assumption, and completes the proof. $\square$

Problem 3. Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. Show that for any $x, y \in (a, b)$, if k is a number between $f'(x)$ and $f'(y)$, then there is some $c \in (x, y)$ such that $f'(c) = k$. **Note.** This means the a derivative function has the intermediate value property. This should help you construct a function that is not continuous but has the intermediate value property!

Proof. (Different from the proof outlined in Apostol's Exercise 4.15.10.) Let $x, y \in (a, b)$. Assume $x < y$, $f'(x) < f'(y)$ and $k \in (f'(x), f'(y))$. Let $g(t) = kt - f(t)$. Then, g is a continuous function on $[a, b]$, and therefore $[x, y]$, and attains its global maximum on $[x, y]$ at some $c \in [x, y]$. We will show that c is neither x nor y, in which case $g'(c) = 0$ or $f'(c) = k$.
Note that g is differentiable at x. Moreover, $g'(x) = k - f'(x) > 0$. Thus, by the $\varepsilon - \delta$ definition of differentiability, there is a $\delta > 0$ (corresponding to $\varepsilon = g'(x)/2$) such that for all $t \in (x, x + \delta)$,

$$0 < g'(x) - \frac{g'(x)}{2} < \frac{g(t) - g(x)}{t - x}.$$

Since the denominator of the right-hand term is positive, the numerator must be positive, so $g(t) > g(x)$ for all $t \in (x, x+\delta$. Thus, x is not a point of global maximum for g on $[x, y]$. A similar argument eliminates y as a point of global max. for g on $[x, y]$. Thus, $c \in (x, y)$ and we are done.
The case $f'(x) > f'(y)$ can be handled similarly. □

Problem 4. Show that

$$1 + \frac{1}{2}x - \frac{1}{8}x^2 \leq \sqrt{1+x} \leq 1 + \frac{1}{2}x$$

for all $x > 0$.

Proof. Since polynomials are infinitely differentiable on $\mathbb{R}$ and x^r, $r \in \mathbb{Q}$, is infinitely differentiable on $(0, \infty)$, we have that $(1+x)^{1/2}$ is infinitely differentiable on $(-1, \infty)$. The 1^{st} and 2^{nd} Taylor polynomials of $f(x) = \sqrt{1+x}$ at $c = 0$ are

$$\begin{aligned} P_1^0(x) &= 1 + \frac{x}{2} \\ P_2^0(x) &= 1 + \frac{x}{2} - \frac{x^2}{8}. \end{aligned}$$

Thus, by Taylor's theorem, for any $x > 0$, there exist $b_x, c_x \in (0, x)$ such that

$$\begin{aligned} f(x) &= P_1^0(x) + \frac{f''(b_x)}{2!}x^2 = 1 + \frac{x}{2} - \frac{1}{8}(1+b_x)^{-3/2}x^2 < 1 + \frac{x}{2}, \\ f(x) &= P_2^0(x) + \frac{f'''(c_x)}{3!}x^3 = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{1}{16}(1+c_x)^{-5/2}x^3 > 1 + \frac{x}{2} - \frac{x^2}{8}. \end{aligned}$$

□

Problem 5. Locate and classify all the points of local extrema of the function

$$f(x) = x|x^2 - 12|$$

on the domain $[-2, 3]$.

Solution. Note that $f(x) = x(12 - x^2)$ for $x \in [-2, 3]$. Thus, f is differentiable on $(-2, 3)$ and $f'(x) = 12 - 3x^2$. Thus, $f'(x) = 0$ in $(-2, 3)$ precisely when $x = 2$. So the potential points of local extrema are $\{-2, 2, 3\}$.
$\boxed{x = -2}$ For $-2 < x < -1$, we have that $1 < x^2 < 4$, or $9 > 12 - 3x^2 > 0$. Thus, $f' > 0$ on $(-2, -1)$, and $f(x) \geq f(-2)$ for all $x \in (-2, -1)$. Thus, -2 is a point of local minima for f on $[-2, 3]$.
A similar analysis will yield that $x = 2$ is a point of local maximum, and 3 is neither.

$*****$The following problems may require more time than the rest.$*****$

Solutions to these problems will not be provided. These problems are for "fun" and you can continue to try them through the semester, or even later.

Problem 6 (Proof of Taylor's Theorem). Recall the following theorem. Let $f : (a, b) \to \mathbb{R}$ be $(n+1)$-times differentiable on (a, b). Let $x_0 \in (a, b)$. Then, for any $x \in (a, b)$, there exists a c_x between x and x_0 such that

$$f(x) = P_n^{x_0}(x) + \frac{f^{(n+1)}(c_x)}{(n+1)!}(x - x_0)^{n+1}.$$

Brainstorming. WLOG assume $x < x_0$. Set

$$K_n = (n+1)! \frac{f(x) - P_n^{x_0}(x)}{(x - x_0)^{n+1}}.$$

We will try to apply MVT to an appropriate function $G : [x, x_0] \to \mathbb{R}$ that satisfies two conditions:

(i) $G(x) = G(x_0)$

(ii) $G'(c) = 0$ implies that

$$f^{(n+1)}(c) = K_n. \tag{1}$$

In class, we observed that the naive idea of $G(t) = f^{(n)}(t) - tK_n$ does not satisfy (i).

Let us attempt a new G for the case $n = 1$. Remember that x is fixed and the variable of differentiation is t in the below argument. Note that we can write (1) as

$$\begin{aligned} & f''(c) = K_1 \\ \iff\ & (x - c)f''(c) = (x - c)K_1 \\ \iff\ & \Big[f(t) + (x - t)f'(t)\Big]'_{t=c} = -K_1\left[\frac{(x-t)^2}{2}\right]'_{t=c} \\ \iff\ & \left[P_1^t(x) + (x - t)^2\frac{K_1}{2!}\right]'_{t=c} = 0 \end{aligned}$$

(a) Use the function $G(t) = P_1^t(x) + \frac{(x-t)^2 K_2}{2!}$ to prove Taylor's theorem in the case of $n = 1$.

(b) By modifying G suitably for each n, prove Taylor's theorem. *Hint.* It may help to try $n = 2$ before attempting the general case.

Problem 7. Let $f : \mathbb{R} \to \mathbb{R}$ and $c \in \mathbb{R}$. Consider the following "definitions". In each case, determine whether it is equivalent to the definition given in class. If yes, provide a proof. If not, provide an example.

(*i*) We say that f admits a limit as x approaches c if for every $\varepsilon > 0$, there exists an $L \in \mathbb{R}$ and a $\delta > 0$ such that for every $x \in N_\delta(c) \setminus \{c\}$, we have that

$$|f(x) - L| < \varepsilon.$$

(*ii*) We say that f admits a limit as x approaches c if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $x \in N_\delta(c) \setminus \{c\}$ there exists an $L \in \mathbb{R}$ such that

$$|f(x) - L| < \varepsilon.$$

(*iii*) We say that f admits a limit as x approaches c if there exists an $L > 0$ and a $\delta > 0$, such that for every $\varepsilon > 0$, whenever $x \in N_\delta(c) \setminus \{c\}$, we have that

$$|f(x) - L| < \varepsilon.$$

Note. The text in red marks the main departure from the original definition.