UM 101 HOMEWORK ASSIGNMENT 9

Posted on December 29, 2022
(NOT FOR SUBMISSION)

- These problems are for self-study.
- Some of these problems will be discussed at the next tutorial.
- A 15-minute quiz based on the topics of this assignment will be conducted at the end of the tutorial section.

Problem 1. Recall that if $s : [a, b] \to \mathbb{R}$ is a step function with respect to a partition $\mathcal{P} = \{x_0 < x_1 < \cdots < x_n\}$ of $[a, b]$, i.e., there exist $s_1, ..., s_n \in \mathbb{R}$ such that

$$s(x) = s_j, \text{ for } x \in (x_{j-1}, x_j),$$

then we define $\int_a^b s(x)dx = \sum_{j=1}^n s_j(x_j - x_{j-1})$. Let us denote this quantity by $\int_{\mathcal{P}} s(x)dx$ instead, for the purpose of this problem. Show that if s is also a step function with respect to a partition $\mathcal{Q}$ of $[a, b]$, then $\int_{\mathcal{P}} s(x)dx = \int_{\mathcal{Q}} s(x)dx$.

Hint. First, show that if s is a step function with respect to a partition $\mathcal{P}$, then it is a step function with respect to any refinement $\mathcal{P}'$ of $\mathcal{P}$. Next, show that $\int_{\mathcal{P}} s(x)dx = \int_{\mathcal{P}'} s(x)dx$.

Problem 2. Compute the following quantities using the definition of $\int_a^b s(x)dx$ given in Lecture 26 (or Problem 1 above). Here, $[x]$ denotes the greatest integer that is less than or equal to x, for any $x \in \mathbb{R}$.

(a) $\int\limits_{-1}^{2} \left([x - \frac{1}{2}] + [x]\right) dx.$

(b) $\int\limits_{1}^{9} [\sqrt{x}]dx.$

Problem 3. Let $s : [a, b] \to \mathbb{R}$ be a step function. Show that s is Riemann integrable, and the two ways of computing $\int_a^b s(x)dx$ yield the same value.

For the rest of the assingment, you may use freely use Theorems 1.16-1.20 from Apostol.

Problem 4. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. For any $[c, d] \subseteq [a, b]$, show that f restricted to the domain $[c, d]$ is Riemann integrable on $[c, d]$.

Problem 5. Exercise 25 from Section 1.26 in Apostol.

Problem 6. Let $f : [a, b] \to \mathbb{R}$ be a continuous and nonnegative function, i.e., $f(x) \geq 0$ for all $x \in [a, b]$. Suppose f is Riemann integrable (we are yet to show this for continuous functions, in general) and $\int_a^b f(x)dx = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.

Problem 7. Show that following definition is equivalent to the definition given in class. A bounded function $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* on $[a, b]$ if, there exists an $I \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exist step functions $s_\varepsilon, t_\varepsilon : [a, b] \to \mathbb{R}$ such that $s_\varepsilon \leq f$, $t_\varepsilon \geq f$,

$$\int_a^b s_\varepsilon(x)dx > I - \varepsilon$$

and

$$\int_a^b t_\varepsilon(x)dx < I + \varepsilon.$$

In this case, we say that the integral of f over $[a, b]$ is I.