UM 101 HOMEWORK ASSIGNMENT 9
SKETCH OF SOLUTIONS

Problem 1. Recall that if $s : [a, b] \to \mathbb{R}$ is a step function with respect to a partition $\mathcal{P} = \{x_0 < x_1 < \cdots < x_n\}$ of $[a, b]$, i.e., there exist $s_1, ..., s_n \in \mathbb{R}$ such that

$$s(x) = s_j, \text{ for } x \in (x_{j-1}, x_j),$$

then we define $\int_a^b s(x)dx = \sum_{j=1}^n s_j(x_j - x_{j-1})$. Let us denote this quantity by $\int_{\mathcal{P}} s(x)dx$ instead, for the purpose of this problem. Show that if s is also a step function with respect to a partition $\mathcal{Q}$ of $[a, b]$, then $\int_{\mathcal{P}} s(x)dx = \int_{\mathcal{Q}} s(x)dx$.

Hint. First, show that if s is a step function with respect to a partition $\mathcal{P}$, then it is a step function with respect to any refinement $\mathcal{P}'$ of $\mathcal{P}$. Next, show that $\int_{\mathcal{P}} s(x)dx = \int_{\mathcal{P}'} s(x)dx$.

Proof of Hint: We can prove this by induction as follows. We define

$\mathbf{T(n)}$:= If $\mathcal{P}_n$ is a refinement of $\mathcal{P}$ with n additional points, then s is also a step function with respect to $\mathcal{P}_n$ and $\int_{\mathcal{P}_n} s(x)dx = \int_{\mathcal{P}} s(x)dx$.

$\boxed{\mathbf{n = 1}}$ Let $\mathcal{P}_1 = \mathcal{P} \cup \{y\}$ where $y \neq x_i$ for $0 \le i \le n$. There is a k, $0 \le k \le n$ such that $y \in (x_{k-1}, x_k)$. We have $\mathcal{P}_1 = \{x_0 < x_1 < \cdots < x_{k-1} < y < x_k < \cdots < x_n\}$. So, if we define a function

$$s'(x) = \begin{cases} s_j, & x \in (x_{j-1}, x_j), j \neq k, \\ s_k, & x \in (x_{k-1}, y), \\ s_k, & x \in (y, x_k). \end{cases}$$

Then s' is a step function for $\mathcal{P}_1$ and $s(x) = s'(x)$ for $x \in (x_{j-1}, x_j) \cup (x_{k-1}, y) \cup (y, x_k)$. So, s is a step function for $\mathcal{P}_1$. Also

$$\begin{aligned} \int_{\mathcal{P}_1} s(x)dx &= \sum_{j=1}^{k-1} s_j(x_j - x_{j-1}) + \sum_{j=k+1}^{n} s_j(x_j - x_{j-1}) + s_k(y - x_{k-1}) + s_k(x_k - y) \\ &= \sum_{j=1}^{k-1} s_j(x_j - x_{j-1}) + \sum_{j=k+1}^{n} s_j(x_j - x_{j-1}) + s_k(x_k - x_{k-1}) \\ &= \sum_{j=1}^{n} s_j(x_j - x_{j-1}) = \int_{\mathcal{P}} s(x)dx \end{aligned}$$

Hence, $T(1)$ is true. Now if $T(k)$ is true for some k then by using the similar argument above, $T(k+1)$ is also true. By the Principle of Mathematical induction, $T(n)$ holds for all $n \in \mathbb{N}$.

Since for any refinement $\mathcal{P}'$ of $\mathcal{P}$, there is an $m \in \mathbb{N}$ such that $\mathcal{P}' = \mathcal{P}_m$. So, s is a step function w.r.t. $\mathcal{P}'$ and $\int_{\mathcal{P}'} s(x)dx = \int_{\mathcal{P}} s(x)dx$.

Proof of Problem 1. Now for any partition $\mathcal{Q}$ of $[a, b]$ for which s is a step function, $\mathcal{P} \cup \mathcal{Q}$ is the common refinement of $\mathcal{P}$ and $\mathcal{Q}$. So, from the above proof,

$$\int_{\mathcal{P}} s(x)dx = \int_{\mathcal{P}\cup\mathcal{Q}} s(x)dx = \int_{\mathcal{Q}} s(x)dx$$

Problem 2. Compute the following quantities using the definition of $\int_a^b s(x)dx$ given in Lecture 26 (or Problem 1 above). Here, $[x]$ denotes the greatest integer that is less than or equal to x, for any $x \in \mathbb{R}$.

(a) $\int\limits_{-1}^{2} \left([x - \frac{1}{2}] + [x]\right) dx$. Solution. (a) $s(x) = [x - \frac{1}{2}] + [x]$ is written as a step function w.r.t. the partition $\{-1, -\frac{1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, 2\}$ as

$$s(x) = \begin{cases} -2 - 1, \; x \in (-1, -\frac{1}{2}), \\ -1 - 1, \; x \in (-\frac{1}{2}, 0), \\ -1 + 0, \; x \in (0, \frac{1}{2}), \\ 0 + 0, \; x \in (\frac{1}{2}, 1), \\ 0 + 1, \; x \in (1, \frac{3}{2}), \\ 1 + 1, \in (\frac{3}{2}, 2) \end{cases}$$

Thus, by the formula given in class, $\int\limits_{-1}^{2} \left([x - \frac{1}{2}] + [x]\right) dx = -3.\frac{1}{2} + -2.\frac{1}{2} - 1.\frac{1}{2} + 0.\frac{1}{2} + 1\frac{1}{2} + 2\frac{1}{2} = -\frac{3}{2}$.

(b) $\int\limits_{1}^{9} [\sqrt{x}]dx$.

Similar to (a).

Problem 3. Let $s : [a, b] \to \mathbb{R}$ be a step function. Show that s is Riemann integrable, and the two ways of computing $\int_a^b s(x)dx$ yield the same value.

Proof. We need to show that s is Riemann integrable, i.e $\underline{I}(s) = \overline{I}(s)$. We will show that $\underline{I}(s) = \overline{I}(s) = \sum_{j=1}^{n} s_j(x_j - x_{j-1})$. Call $\sum_{j=1}^{n} s_j(x_j - x_{j-1}) = k.$, where $\mathcal{P} = \{x_0 < \cdots < x_n\}$ is a partition of $[a, b]$ w.r.t. which s is a step function.

We will show that $\underline{I}(s) = \sup\left\{\int_a^b \widetilde{s}(x)dx : \widetilde{s} \in S_s\right\} = k$. If $\widetilde{s} \in S_s$, then $\widetilde{s}(x) \le s(x)$ for all $x \in [a, b]$. By the comparison theorem for step functions, $\int_a^b \widetilde{s}(x)dx \le \int_a^b s(x)dx = k$. Thus k is an upper bound of the set $\left\{\int_a^b \widetilde{s}(x)dx : \widetilde{s} \in S_s\right\}$. Let $\widetilde{k} < k$. We need to find an $\widetilde{s} \in S_s$

such that s$\int_a^b \widetilde{s}(x)dx > \widetilde{k}$. Take $\widetilde{s} = s$, since s itself is a step function and, hence, a member of S_s. Thus, $k = \underline{I}(s)$. Similarly show that $\overline{I}(s) = \inf\left\{\int_a^b t(x)dx : t \in T_s\right\} = k$.

For the rest of the assignment, you may use freely use Theorems 1.16-1.20 from Apostol.

Problem 4. Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. For any $[c, d] \subseteq [a, b]$, show that f restricted to the domain $[c, d]$ is Riemann integrable on $[c, d]$.

Proof. **Note. One is proving Problem 7 on the way!** Let $\varepsilon > 0$. By the definition of $\underline{I}(f)$ and $\overline{I}(f)$, there exist $s_\varepsilon \in S_f$ and $t_\varepsilon \in T_f$ such that

$$\underline{I}(f) - \varepsilon < \int_a^b s_\varepsilon(x)dx$$

and

$$\overline{I}(f) + \varepsilon > \int_a^b t_\varepsilon(x)dx.$$

Thus,

$$\int_a^b t_\varepsilon(x)dx - \int_a^b s_\varepsilon(x)dx < \overline{I} + \varepsilon - (\underline{I}(f) - \varepsilon) = 2\varepsilon$$

as $\overline{I}(f) = \underline{I}(f)$ (f is R.I.). By Theorem 1.16 (Linearity w.r.t Integral) we get that

$$\int_a^b (t_\varepsilon - s_\varepsilon)(x)dx < 2\varepsilon.$$

We denote $f\big|_{[c,d]}$ by $\widetilde{f}$. Consider the step functions

$$\widetilde{t_\varepsilon}(x) = \begin{cases} t_\varepsilon(x), & x \in [c, d], \\ 0 \text{ otherwise.} \end{cases}$$

and

$$\widetilde{s_\varepsilon}(x) = \begin{cases} s_\varepsilon(x), & x \in [c, d], \\ 0 \text{ otherwise.} \end{cases}$$

Observe that $\widetilde{s_\varepsilon} \in S_{\widetilde{f}}$, $\widetilde{t_\varepsilon} \in T_{\widetilde{f}}$, and $0 \le \widetilde{t_\varepsilon}(x) - \widetilde{s_\varepsilon}(x) \le t_\varepsilon(x) - s_\varepsilon(x)$ for all $x \in [a, b]$. Thus, by Theorem 1.20 (comparison)

$$0 \le \int_a^b \widetilde{t_\varepsilon}(x) - \widetilde{s_\varepsilon}(x)dx < 2\varepsilon.$$

By Theorem 1.17 (additivity w.r.t the interval of integration) and the fact that both $\widetilde{s_\varepsilon}$ and $\widetilde{t_\varepsilon}$ vanish outside $[c, d]$, we have that

$$0 \le \int_c^d \left(\widetilde{t_\varepsilon}(x) - \widetilde{s_\varepsilon}(x)\right) dx < 2\varepsilon.$$

Now, since $\underline{I}\left(\widetilde{f}\right) \geq \int_c^d \widetilde{s}_\varepsilon(x)dx$ and $\overline{I}\left(\widetilde{f}\right) \leq \int_c^d \widetilde{t}_\varepsilon(x)dx$, we have that

$$0 \leq \overline{I}\left(\widetilde{f}\right) - \underline{I}\left(\widetilde{f}\right) \leq \int_c^d \left(\widetilde{t}_\varepsilon(x) - \widetilde{s}_\varepsilon(x)\right) dx < 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we obtain the integrability of $\widetilde{f}$. □

Problem 5. Exercise 25 from Section 1.26 in Apostol.

You can use Theorem 1.17 and Theoremm 1.19 (for k=-1) from Apostol, and the fact that

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

Problem 6. Let $f : [a, b] \to \mathbb{R}$ be a continuous and nonnegative function, i.e., $f(x) \geq 0$ for all $x \in [a, b]$. Suppose f is Riemann integrable (we are yet to show this for continuous functions, in general) and $\int_a^b f(x)dx = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.

Suppose there is a $c \in [a, b]$ such that $f(c) > 0$. By the continuity of f, for $\varepsilon = \frac{f(c)}{2}$, there is a $\delta > 0$ such that for $x \in (c - \delta, c + \delta)$, $|f(x) - f(c)| < \frac{f(c)}{2} \Rightarrow f(c) - \frac{f(c)}{2} < f(x) < f(c) + \frac{f(c)}{2} \Rightarrow 0 < \frac{f(c)}{2} < f(x)$. So, $f(x) > 0$ for $x \in (c - \delta, c + \delta)$.

Now, f is continuous and > 0 on $[c - \delta/2, c + \delta/2]$. By a previous HW problem, $m = \min_{[c-\delta/2, c+\delta/2]} f > 0$. Define a step function s as

$$s(x) = \begin{cases} m, & x \in (c - \delta/2, c + \delta/2), \\ 0, & \text{otherwise.} \end{cases}$$

Then, $s \leq f$ on $[a, b]$ and with the partition $\mathcal{P} = \{a, c - \delta/2, c + \delta/2, b\}$, $\int_a^b s(x)dx = m\delta > 0$ So, $0 < \int_a^b s(x)dx \leq \underline{I}(f) = \int_a^b f(x)dx$ which contradicts the fact that $\int_a^b f(x)dx = 0$.

Problem 7. Show that following definition is equivalent to the definition given in class. A bounded function $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* on $[a, b]$ if, there exists an $I \in \mathbb{R}$ such that for every $\varepsilon > 0$, there exist step functions $s_\varepsilon, t_\varepsilon : [a, b] \to \mathbb{R}$ such that $s_\varepsilon \leq f$, $t_\varepsilon \geq f$,

$$\int_a^b s_\varepsilon(x)dx > I - \varepsilon$$

and

$$\int_a^b t_\varepsilon(x)dx < I + \varepsilon.$$

In this case, we say that the integral of f over $[a, b]$ is I.

R.I.$\Rightarrow$ ε-characterization. See the first few lines of the solution to Problem 4.

ε-characterization $\Rightarrow$ R.I. See the last few lines of the solution to Problem 4.