# UM 101 HOMEWORK ASSIGNMENT 10

Posted on January 5, 2023
(NOT FOR SUBMISSION)

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- These problems are for self-study.
- Some of these problems will be discussed at the next tutorial.
- A 15-minute quiz based on the topics of this assignment will be conducted at the end of the tutorial section.

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**Problem 1.** Let $A \subseteq \mathbb{R}$ be a non-empty subset. Let $f : A \to \mathbb{R}$ be a *Lipschitz* function on $A$, i.e., for some $M > 0$,

$$|f(x) - f(y)| \leq M|x - y|, \quad \forall x, y \in A.$$

Show that $f$ is uniformly continuous on $A$. Produce a uniformly continuous function on $[0, \infty)$ that is not Lipschitz on $[0, \infty)$.

**Problem 2.** Show that the following function is Riemann integrable on $[0, 1]$.

$$f(x) = \begin{cases} x - x^2, & x \in [0,1] \setminus \{1/3\}, \\ 0, & x = 1/3. \end{cases}$$

Generalize your proof to show that every bounded function that is continuous at all but finitely many points in an interval $[a, b]$ is Riemann integrable on $[a, b]$. **Hint.** Use the $\epsilon$-characterization of Riemann integrability.

**Problem 3.** Show that the following bounded function is not Riemann integrable on $[0, 1]$.

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \cap [0,1], \\ 0, & x = [0,1] \setminus \mathbb{Q}. \end{cases}$$

**Problem 4.** Argue (possibly citing earlier problems from this assignment) that the function

$$f(x) = \begin{cases} 2x \cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is Riemann integrable on $[-1, 1]$. Find the function (try to guess what $F$ would be, and then justify your answer using a theorem)

$$F(x) = \int_{-1}^{x} f(x)dx, \quad x \in [-1, 1].$$

**Note.** This problem demonstrates that even though $\lim_{x \to 0} f(x)$ does not exist (i.e., the discontinuity of $f$ at $x = 0$ is not "removable"), $F$ can be differentiable at $x = 0$.

**Problem 5.** Solve Problems 14, 15, 16, 17, 19, 20 and 22 from Section 5.5 of Apostol.