UM 101 HOMEWORK ASSIGNMENT 10
SKETCH OF SOLUTIONS

Problem 1. Let $A \subseteq \mathbb{R}$ be a non-empty subset. Let $f : A \to \mathbb{R}$ be a *Lipschitz* function on A, i.e., for some $M > 0$,

$$|f(x) - f(y)| \leq M|x - y|, \quad \forall x, y \in A.$$

Show that f is uniformly continuous on A. Produce a uniformly continuous function on $[0, \infty)$ that is not Lipschitz on $[0, \infty)$.

Solution. To show uniform continuity of f, for any given $\epsilon > 0$, you may choose $\delta = \frac{\epsilon}{M}$.

Example of a uniformly continuous but non-Lipschitz function Let $f(x) = \sqrt{x}$. First, we prove that f is uniformly continuous on $[0, \infty)$. Let $\varepsilon > 0$. Fix an $a > 0$. Since $[0, a]$ is a closed and bounded interval, f is uniformly continuous on $[0, a]$. Thus, there exists a $\delta_1 > 0$ such that for $x, y \in [0, a]$ and $|x - y| < \delta_1$, we have

$$|\sqrt{x} - \sqrt{y}| < \frac{\epsilon}{2}.$$

Now, if $x, y \in [a, \infty)$, then

$$|\sqrt{x} - \sqrt{y}| = \frac{|x - y|}{|\sqrt{x} + \sqrt{y}|} = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} \leq \frac{|x - y|}{2\sqrt{a}}.$$

Let $\delta_2 = \sqrt{a}\epsilon$. Then, for any $x, y \in [a, \infty)$ such that $|x - y| < \delta_2$, we have that

$$|\sqrt{x} - \sqrt{y}| < \frac{\varepsilon}{2}.$$

Let

$$\delta = \min\{\delta_1, \delta_2\}.$$

Let $x, y \in [0, \infty)$ such that $|x - y| < \delta$. If $x, y \in [0, a]$ or $x, y \in [a, \infty)$, we have that $|\sqrt{x} - \sqrt{y}| < \varepsilon/2 < \varepsilon$. Thus, we assume (WLOG) that $x \in [0, a]$ and $y \in [a, \infty)$. Then, $|x - a| \leq |x - y| < \delta < \delta_1$ and $|a - y| \leq |x - y| < \delta \leq \delta_2$. Thus,

$$|\sqrt{x} - \sqrt{(y)} \leq |\sqrt{x} - \sqrt{a}| + |\sqrt{a} - \sqrt{y}| < 2\frac{\varepsilon}{2} = \varepsilon.$$

Now we show that f is not Lipschitz on $[0, \infty)$. For any real $M > 0$, Let $x = 0$ and $y = \frac{1}{(M+1)^2}$. Then $|\sqrt{x} - \sqrt{y}| = |0 - \frac{1}{\sqrt{(M+1)^2}}| = \frac{1}{M+1}$. Since $M + 1 > M \Rightarrow 1 > \frac{M}{M+1} \Rightarrow$

$\frac{1}{M+1} > \frac{M}{(M+1)^2}$. So, we have

$$\frac{1}{M+1} = |\sqrt{x} - \sqrt{y}| > \frac{M}{(M+1)^2} = M|0 - \frac{1}{(M+1)^2}| = M|x-y|.$$

Problem 2. Show that the following function is Riemann integrable on $[0,1]$.

$$f(x) = \begin{cases} x - x^2, & x \in [0,1] \setminus \{1/3\}, \\ 0, & x = 1/3. \end{cases}$$

Generalize your proof to show that every bounded function that is continuous at all but finitely many points in an interval $[a,b]$ is Riemann integrable on $[a,b]$. **Hint.** Use the ϵ-characterization of Riemann integrability.

General Proof. Let $f : [a,b] \to \mathbb{R}$ be a bounded function which is continuous everywhere except at some $c \in [a,b]$.

$\boxed{\text{Case 1. } c = b.}$ Let $M > 0$ be such that $|f(x)| \leq M$ for all $x \in [a,b]$. Let $\epsilon > 0$. Sset

$$\delta = \min\left\{\frac{\varepsilon}{4M}, b-a\right\}.$$

Then, we have that

$$[a,b] = [a, b-\delta] \cup [b-\delta, b].$$

Note that f is R.I. on $[a, b-\delta]$ owing to continuity. Thus, by the ε-char. of R.I., there are step functions $t_\epsilon, s_\epsilon : [a, b-\delta] \to \mathbf{R}$ such that $t_\epsilon \geq f,\ s_\epsilon \leq f$ on $[a, b-\delta]$ and

$$\int_a^{b-\delta} t_\varepsilon(x)dx - \int_a^{b-\delta} s_\varepsilon(x)dx < \frac{\varepsilon}{2}.$$

Now define

$$\widetilde{s}_\varepsilon = \begin{cases} s_\varepsilon(x), & x \in [a, b-\delta], \\ -M, & x \in (b-\delta, b], \end{cases} \qquad \widetilde{t}_\varepsilon = \begin{cases} t_\varepsilon(x), & x \in [a, b-\delta], \\ M, & x \in (b-\delta, b]. \end{cases}$$

Then, $\widetilde{t}_\varepsilon \geq f$ and $\widetilde{s}_\varepsilon \leq f$ on $[a,b]$ and

$$\begin{aligned} \int_a^b \left(\widetilde{t}_\varepsilon(x) - \widetilde{s}_\varepsilon(x)\right) dx &= \int_a^{b-\delta} \left(t_\varepsilon(x) - s_\varepsilon(x)\right) dx + \int_{b-\delta}^b (M - (-M))\, dx \\ &< \frac{\varepsilon}{2} + 2M\frac{\varepsilon}{4M} = \varepsilon. \end{aligned}$$

Hence by ε-characterization of Riemann integrability, f is Riemann integrable on $[a,b]$.

Case 2. $c = a$. Consider $g(x) = f(-x)$ on $[-b, -a]$. Then, g is continuous everywhere except, possibly, at $-a$. By the previous case, g is R.I. on $[-b, -a]$. By Theorem 1.19 from Apostol ($k = -1$), f is R.I. on $[a, b]$

Case 2. $a < c < b$. By the previous two cases, f is R.I. on $[a, c]$ and $[c, b]$. The claim now follows from Theorem 1.17 in Apostol.

We settle the general case by induction. Let $P(k)$ denote the statement that a function on a closed and bounded interval I which is discontinuous at only k points in I is R.I. on I. We have already shown $P(1)$. Suppose $P(k)$ is true. Let $I = [a, b]$ and $f : [a, b] \to \mathbb{R}$ be a function that is discontinuous precisely on $\{c_1 < \cdots < c_{k+1}\} \in [a, b]$. Let $d \in (c_k, c_{k+1})$. By $P(k)$, f is R.I. on $[a, d]$, and by $P(1)$, f is R.I. on $[d, b]$, thus by the additivity of the intervals of integration, f is R.I. on $[a, b]$. Thus, by PMI, the general claim holds.

Problem 3. Show that the following bounded function is not Riemann integrable on $[0, 1]$.

$$f(x) = 1, \quad x \in \mathbb{Q} \cap [0, 1],$$
$$0, \quad x = [0, 1] \setminus \mathbb{Q}.$$

Proof. We need to show that f is not Riemann Integrable on [0,1]. We will do so using the negation of the ϵ-characterization of Riemann integrability.

ϵ-characterization of Riemann integrability:

A bounded function $f : [a, b] \to \mathbb{R}$ is said to be *Riemann integrable* on $[a, b]$ if, for every $\varepsilon > 0$, there exist step functions $s_\varepsilon, t_\varepsilon : [a, b] \to \mathbb{R}$ such that $s_\varepsilon \le f$, $t_\varepsilon \ge f$,

$$\int_a^b t_\varepsilon(x)dx - \int_a^b s_\varepsilon(x)dx < \varepsilon.$$

So we need to show that there exists an $\epsilon > 0$ such that for any two step functions $s, t : [a, b] \to \mathbb{R}$ such that $s \le f$, $t \ge f$,

$$\int_a^b t(x)dx - \int_a^b s(x)dx \ge \varepsilon.$$

We choose $\epsilon = 1$. Now if $s : [a, b] \to \mathbb{R}$ is a step function with partition $P = \{0 = x_0 < ... < x_n = 1\}$ such that $s \le f$, then $s(x) = s_k$ for $x \in (x_{k-1}, x_k)$, where each $s_k \le 0$ (Why?). Therefore $\int_a^b s(x)dx \le 0$. Also if $t : [a, b] \to \mathbb{R}$ is a step function with partition $P = \{0 = y_0 < ... < y_m = 1\}$ such that $t \ge f$, then $t(x) = t_k$ for $x \in (y_{k-1}, y_k)$, where each $t_k \ge 1$. (Why?) Therefore $\int_a^b t(x)dx \ge 1$. Combining the obtained inequalities we get that

$$\int_a^b t(x)dx - \int_a^b s(x)dx \ge 1 - 0 = \varepsilon.$$

And we are done.

Problem 4. Argue (possibly citing earlier problems from this assignment) that the function

$$f(x) = \begin{cases} 2x\cos\left(\frac{1}{x}\right) + \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0, \end{cases}$$

is Riemann integrable on $[-1, 1]$. Find the function (try to guess what F would be, and then justify your answer using a theorem)

$$F(x) = \int_{-1}^{x} f(x)dx, \quad x \in [-1, 1].$$

Note. This problem demonstrates that even though $\lim_{x\to 0} f(x)$ does not exist (i.e., the discontinuity of f at $x = 0$ is not "removable"), F can be differentiable at $x = 0$.

Proof. As cos and sin are bounded functions, one can see that f is bounded on $[-1, 1]$. As f has only one point of discontinuity (i.e., 0) in $[-1, 1]$, by Problem 2, we have that f is R.I. on $[-1, 1]$.

Now consider the function $G(x) = x^2\cos(\frac{1}{x})$, $x \neq 0$ and $F0) = 0$. Check that $G'(x) = f(x)$ for all $x \in \mathbb{R}$. Thus, f admits an antiderivative G on the whole of $\mathbb{R}$, and in particular on the open interval $(-2, 2)$. Now let $x \in [-1, 1] \subset (-2, 2)$. As f is R.I. on $[-1, 1]$ and $[-1, x] \subset [-1, 1]$, therefore $f\Big|_{[-1,x]}$ is R.I. Thus, by The Second Fundamental Theorem of Calculus we have that

$$\int_{-1}^{x} f(t)dt = G(x) - G(-1).$$

Now set $F(x) = G(x) - G(-1)$ $x \in [-1, 1]$.

Problem 5. Solve Problems 14, 15, 16, 17, 19, 20 and 22 from Section 5.5 of Apostol.

Hint for Problem 14. Let $a < b$ be real numbers such that $0, \frac{\pi}{4} \in (a, b)$. Since f is continuous everywhere, by the first FTOC, $F(x) = \int_0^x f(t)dt$ is differentiable on (a, b) and $F'(x) = f(x)$ for all $x \in (a, b)$. Thus, we may differentiate

$$F(x) = \int_0^x f(t)dt = \frac{-1}{2} + x^2 + x\sin 2x + \frac{1}{2}\cos 2x$$

to get that $f(x) = 2x + 2x\cos 2x$. Now calculate and see that $f(\frac{\pi}{4}) = \frac{\pi}{2}$ and $f'(\frac{\pi}{4}) = 2 - \pi$.

Hint for Problem 15. One can guess f by assuming that f must be continuous everywhere and differentiating the R.H.S. of the given formula. This gives $f(x) = -\sin x$. From computations done in class, we have that $\int_c^x -\sin(t)dt = \cos x - \cos(c)$, which is given to be equal to $\cos(x) - 1/2$. Thus, $c = \frac{\pi}{3}$.

Hint for Problem 16. Same approach as Problem 15.

Hint for Problem 17. We are given that f is continuous everywhere. Let us choose $a, b \in \mathbb{R}$ with $a < b$ such that $0, 1, x \in (a, b)$. We are given that f is continuous everywhere, hence it is continuous on $[a, b]$, hence $f\Big|_{[a,b]}$ is Riemann integrable. So, by the first FTOC the function $F(x) = \int_0^x f(t)dt$ is differentiable on (a, b) and $F'(x) = f(x)$ for all $x \in (a, b)$.
Consider the function $g(x) = x^2 f(x)$, it is continuous and for the same reasons as above $g\Big|_{[a,b]}$ is Riemann integrable. So, by the first FTOC, the function $G(x) = -\int_1^x g(t)dt = \int_x^1 g(t)dt$ is differentiable on (a, b) and $G'(x) = -g(x)$ for all $x \in (a, b)$. Now we can differentiate the given equation

$$\int_0^x f(t)dt = -\int_1^x g(t)dt + \frac{x^{16}}{8} + \frac{x^{18}}{9} + c$$

throughout to get

$$f(x) + g(x) = f(x) + x^2 f(x) = 2x^{15} + 2x^{17}.$$

Deduce that $f(x) = 2x^{15}$. Now putting $x = 0$ in the given equation we get that $c = -\frac{1}{9}$.

Hint for Problem 19. Expand $f(x) = \frac{1}{2}\int_0^x (x-t)^2 g(t)dt$ to $f(x) = \frac{1}{2}x^2 \int_0^x g(t)dt - 2x\int_0^x tg(t)dt + \int_0^x t^2 g(t)dt$ and use the same techniques as in Problem 17.

Hint for Problem 20. (a) is straightforward. (b) We are given that $f(x) = \int_0^{x^2} \frac{1}{(1+t^2)^3}$. Since $\frac{1}{(1+t^2)^3}$ is continuous on the whole of $\mathbb{R}$, by thee same argument as in Problem 17, the function $F(x) = \int_0^x \frac{1}{(1+t^2)^3}dt$ is differentiable on $\mathbb{R}$. Since the composition of diff. functions is diff., $f(x) = \int_0^{x^2} \frac{1}{(1+t^2)^3}dt = F(x^2)$ is differentiable on $\mathbb{R}$. Now differentiating throughout we get, $f'(x) = F'(x^2)(2x) = \frac{1}{(1+g(x)^2)^3}2x = \frac{2x}{(1+x^4)^3}$. Do (c) in a similar fashion.

Hint for Problem 22. Same approach as Problem 20.