UM 101 HOMEWORK ASSIGNMENT 13

Posted on January 26, 2023
(NOT FOR SUBMISSION)

- These problems are for self-study.
- Some of these problems will be discussed at the next tutorial.
- A 15-minute quiz based on the topics of this assignment will be conducted at the end of the tutorial section.

Problem 1. In each of the following cases, show that W is a linear subspace of V and compute $\dim(W)$.

(a) $V = \mathbb{R}^3$ and $W = \{(x, y, z) : 2x + 3y = 0 \text{ and } x - y - z = 0\}$.
(b) $V = \mathbb{R}^3$ and $W = \{(x, y, z) : x = 2y = -z\}$.
(c) $V = P_{\leq d}$, the vector space of polynomials of degree at most d, and

$$W = \{f \in P_{\leq d} : f''(0) = 0\}.$$

Problem 2. Let $\mathscr{S}$ denote the vector space of all real sequences. Show that $\mathscr{S}$ is infinite-dimensional. *Hint. Recall the contstraint on linearly independent subsets of a finite-dimensional vector space.*

Problem 3. Let V be a finite-dimensional vector space. Let U be a subspace of V. Show that if $\dim(U) = \dim(V)$, then $U = V$.

Problem 4. Let V be a vector space over $\mathbb{C}$ of dimension n. Show that V is a vector space over $\mathbb{R}$ of dimension $2n$. *Hint. First try the special case $V = \mathbb{C}$.*

Problem 5. Suppose $q_0, ..., q_d \in P_{\leq d}$ are polynomials such that $q_j(0) = 0$ for each $j = 0, ..., d$. Prove that $\{q_0, ..., q_d\}$ is linearly dependent.

Problem 6. Let $U, W \subset V$ be subspaces of a vector space V. Show that $U \cap W$ and the set

$$U + W = \{u + w : u \in U \text{ and } w \in W\}$$

are also subspaces of V. Prove that

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

Hint. For the identity, start with a basis of $U \cap W$.