UM 101 - MIDTERM EXAMINATION

Instructor: Purvi Gupta

December 07, 2022 9:00 am - 12:00 pm (3 hours)

Name of the student : ______________________________

Student number : ______________________________

Instructions

1. This booklet has **22 pages** and **6 questions**. The first question has 4 parts.
2. **Read** every question carefully before attempting to answer it. **Partial credit** is available for partial (but correct) ideas.
3. You may use without proof any results or examples discussed in class or homework assignments. If you do not cite what you are using, you will be penalized.
4. The answer to each question must be written below it. If you run out of space, you may use Pages 20-22, or even attach extra sheets, but these will only be graded if you clearly **tell the reader** to turn to these pages.
5. All the blank spaces can also be used for **scrapwork**.
6. Do not separate any pages from this booklet.

Grading table

Problem #	Points	Score
1	20	
2	8	
3	8	
4	8	
5	8	
6	8	
Total	60	

Problem 1. (*20 points*) For each of the statements below, determine whether it is true or false. If you circle TRUE, you must provide a proof. If you circle FALSE — depending on the statement — you must either provide a counterexample and justify it, or provide a mathematical proof.

(a) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions such that f is continuous on $\mathbb{R}$ and $f \circ g$ is continuous on $\mathbb{R}$. Then, g is continuous on $\mathbb{R}$.

TRUE FALSE

EXTRA SPACE FOR PROBLEM 1 (a).

Problem 1 (True or False) continued.

(*b*) Let $\sum_{n\in\mathbb{N}} a_n$ be an infinite series of nonnegative real numbers such that $\sum_{k\in\mathbb{N}} a_{2k}$ and $\sum_{k\in\mathbb{N}} a_{2k+1}$ converge. Then, $\sum_{n\in\mathbb{N}} a_n$ converges.

TRUE FALSE

EXTRA SPACE FOR PROBLEM 1 (b).

Problem 1 (True or False) continued.

(c) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ is a bounded sequence, then $\{f(a_n)\}_{n\in\mathbb{N}}$ is a bounded sequence.

TRUE FALSE

EXTRA SPACE FOR PROBLEM 1 (c).

Problem 1 (True or False) continued.

(*d*) Let F be the set of continuous functions from $[0,1]$ to $\mathbb{R}$. Let '+' and '·' be the operations on F given by

$$\begin{aligned} f+g &\quad \text{is the function on } [0,1] \text{ given by} \quad (f+g)(x) = f(x)+g(x) \\ f\cdot g &\quad \text{is the function on } [0,1] \text{ given by} \quad (f\cdot g)(x) = f(x)g(x). \end{aligned}$$

Then, $(F,+,\cdot)$ is a field.

TRUE FALSE

EXTRA SPACE FOR PROBLEM 1 (d).

Problem 2. (*8 points*) Determine whether the following series is absolutely convergent, conditionally convergent or divergent. You must justify your answer, clearly stating any tests or theorems that you are using.

$$\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{n+20}.$$

EXTRA SPACE FOR PROBLEM 2.

Problem 3. (*8 points*) The following set is an interval in $\mathbb{R}$:

$$\bigcup_{\substack{n\in\mathbb{N}\\ n\neq 0}} \left(\frac{1}{n}, 1+\frac{1}{n}\right).$$

Determine what interval it is. You must prove your answer, clearly stating any theorems that you are using. **Note. The difference between round and square brackets is crucial in this problem, so please give a legible answer!**

EXTRA SPACE FOR PROBLEM 3.

Problem 4. (*8 points*) Compute the following limit, if it exists. Prove your answer from first principles, i.e., using the ϵ-δ definition of the limit of a function.

$$\lim_{x\to 2} \frac{3x}{x^2-1}.$$

EXTRA SPACE FOR PROBLEM 4.

Problem 5. (*8 points*) Let $f : [0,1] \to \mathbb{R}$ be a continuous function that attains its global maximum on $[0,1]$ at some $c \in (0,1)$. Show rigorously (not just pictorially) that f is **not** one-to-one on $[0,1]$. (*A function $f : A \to \mathbb{R}$ is called one-to-one if: for any $x, y \in A$, if $x \neq y$, then $f(x) \neq f(y)$.*)

EXTRA SPACE FOR PROBLEM 5.

Problem 6. (*8 points*) Prove that for any $r > 0$,

$$\lim_{n \to \infty} \frac{n^2}{(1+r)^n} = 0.$$

You must justify your answer, clearly stating any theorems that you are using.

Note. You are free to use the identity $(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$, $x > 0$, without proof.

EXTRA SPACE FOR PROBLEM 6.

EXTRA SPACE FOR ANSWERS OR ROUGH WORK

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