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Volume Approximations of Strongly Pseudoconvex Domains

Purvi Gupta[1]

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Abstract In convex geometry, the Blaschke surface area measure on the boundary of a convex domain can be interpreted in terms of the complexity of approximating polyhedra. This approach is formulated in the holomorphic setting to establish an alternate interpretation of Fefferman's hypersurface measure on boundaries of strongly pseudoconvex domains in $\mathbb{C}^2$. In particular, it is shown that Fefferman's measure can be recovered from the Bergman kernel of the domain.

Keywords Strongly pseudoconvex domains · Fefferman hypersurface measure · Affine surface area measure · Polyhedral approximations

Mathematics Subject Classification 32T15

1 Introduction

The Fefferman hypersurface measure on the boundary of a $\mathcal{C}^2$-smooth domain $\Omega \subset \mathbb{C}^d$ is the $(2d-1)$-form, σ_Ω, satisfying

$$\sigma_\Omega \wedge d\rho = 4^{\frac{d}{d+1}} M(\rho)^{\frac{1}{d+1}} \omega_{\mathbb{C}^d},$$

where $\omega_{\mathbb{C}^d}$ is the standard volume form on $\mathbb{C}^d$, ρ is a defining function for Ω with $\Omega = \{\rho < 0\}$, and

$$M(\rho) = -\det \begin{pmatrix} \rho & \rho_{\overline{z_k}} \\ \rho_{z_j} & \rho_{z_j \overline{z_k}} \end{pmatrix}_{1 \le j,k \le d}.$$

✉ Purvi Gupta
purvi.gpt@gmail.com

[1] Department of Mathematics, University of Michigan, Ann Arbor, MI 48105, USA

First introduced by Fefferman in [9], this measure is well defined under the added assumption that Ω is strongly pseudoconvex (defined in Sect. 2). Moreover, it does not depend on the choice of ρ and satisfies the following transformation law:

$$F^*\sigma_{F(\Omega)} = |\det \mathrm{J}_{\mathbb{C}}\, F|^{\frac{2d}{d+1}}\sigma_\Omega,$$

where F is a biholomorphism on Ω that is $\mathcal{C}^2$-smooth on $\overline{\Omega}$.

The Fefferman hypersurface measure shares strong connections with the Blaschke surface area measure (explored in [3] and [4], for instance) studied in affine convex geometry. If $K \subset \mathbb{R}^d$ is a $\mathcal{C}^2$-smooth convex body, the Blaschke surface area measure on ∂K is given by

$$\tilde{\sigma}_K = \kappa^{\frac{1}{d+1}} s_{\mathrm{Euc}},$$

where κ and s_{Euc} are the Gaussian curvature function and the Euclidean surface area form on ∂K, respectively. Its resemblance to the Fefferman measure is reflected in the following identity:

$$A^*\tilde{\sigma}_{A(K)} = |\det \mathrm{J}_{\mathbb{R}}\, A|^{\frac{d-1}{d+1}}\tilde{\sigma}_K,$$

where A is an affine transformation of $\mathbb{R}^d$. Since its introduction by Blaschke in [5], several mathematicians have extended the notion of affine surface area to arbitrary convex bodies; see [14] for details. As this measure is invariant under volume-preserving affine maps, it occurs naturally in volume approximations of convex bodies by polyhedra (see [11, Chap. 1.10] for a survey). The first complete asymptotic result was due to Gruber [10] who showed that if $K \subset \mathbb{R}^d$ is a $\mathcal{C}^2$-smooth strongly convex body, then

$$\inf\{\mathrm{vol}(P \setminus K) : P \in \mathcal{P}_n^c\} \sim \frac{1}{2}\,\mathrm{div}_{d-1}\left(\int_{\partial K}\tilde{\sigma}_K\right)^{(d+1)/(d-1)}\frac{1}{n^{2/(d-1)}} \qquad (1.1)$$

as $n \to \infty$, where $\mathcal{P}_n^c$ is the class of all polyhedra that circumscribe K and have at most n facets, and div_{d-1} is a dimensional constant. Ludwig [15] later showed that, if the approximating polyhedra are from $\mathcal{P}_n$, the class of *all* polyhedra with at most n facets, then

$$\inf\{\mathrm{vol}(K \Delta P) : P \in \mathcal{P}_n\} \sim \frac{1}{2}\,\mathrm{ldiv}_{d-1}\left(\int_{\partial K}\tilde{\sigma}_K\right)^{(d+1)/(d-1)}\frac{1}{n^{2/(d-1)}} \qquad (1.2)$$

as $n \to \infty$, where Δ denotes the symmetric difference between sets and ldiv_{d-1} is a dimensional constant. In (1.1) and (1.2), the constants div_{d-1} and ldiv_{d-1} are named after Dirichlet–Voronoi and Laguerre–Dirichlet–Voronoi tilings (see the Appendix), respectively, since these are used to prove the formulae. Later, Böröczky [6] proved both these formulae for all $\mathcal{C}^2$-smooth convex bodies. Similar asymptotics have been obtained using other notions of complexity for a polyhedron—such as the number of vertices.

In [3], Barrett asks whether such relations can be found between the Fefferman hypersurface measure on a pseudoconvex domain and the complexity of approximating analytic polyhedra. An *analytic polyhedron* in Ω is a relatively compact subset that is a union of components of any set of the form

$$P = \{z \in \Omega : |f_j(z)| < 1,\ j = 1, \ldots, n\},$$

where $f_1, \ldots, f_n$ are holomorphic functions in Ω. The natural notion of complexity for an analytic polyhedron, P, is its order, i.e., the number of inequalities that define P. This setup, however, is not suited for our purpose as demonstrated by a result due to Bishop (Lemma 5.3.8 in [12]) which says that any pseudoconvex domain in $\mathbb{C}^d$ can be approximated arbitrarily well (in terms of the volume of the gap) by analytic polyhedra of order at most $2d$. With the help of an example, we indicate where the problem lies. Let $\Omega = \mathbb{D}$ be the unit disk in $\mathbb{C}$. Consider the lemniscate-bound domains

$$P_n := \left\{ z \in \mathbb{D} : \prod_{k=0}^{2n-1} \frac{n}{\pi} \left| (z - \exp(\tfrac{k\pi i}{n}))^{-1} \right| < 1 \right\}.$$

Each P_n has order 1 and satisfies $\{|z| < 1 - \pi/n\} \subset P_n \subset \{|z| < 1 - \sqrt{3}\pi/2n\}$. Thus,

$$\inf\{\mathrm{vol}(\mathbb{D} \setminus P) : P \text{ is an analytic polyhedron of order } 1\} = 0.$$

If we, instead, declare the complexity of P_n to be $2n$, i.e., the number of poles of the function defining P_n, then, since $\lim_{n\to\infty} n \cdot \mathrm{vol}(\mathbb{D} \setminus P_n) \in (0, \infty)$, we can expect results similar to (1.1) and (1.2).

The above example leads us to a special class of polyhedral objects. For any fixed $f \in \mathcal{C}(\overline{\Omega} \times \partial\Omega)$, let $\mathcal{P}_n(f)$ be the collection of all relatively compact sets in Ω of the form

$$P = \left\{ z \in \Omega : |f(z, w^j)| > \delta_j,\ j = 1, \ldots, n \right\},$$

where $w^1, \ldots, w^n \in \partial\Omega$ and $\delta_1, \ldots, \delta_n > 0$. We present a class of functions f for which asymptotic results such as (1.1) and (1.2) can be obtained for domains in $\mathbb{C}^2$.

Theorem 1.1 *Let $\Omega \subset\subset \mathbb{C}^2$ be a $\mathcal{C}^4$-smooth strongly pseudoconvex domain. Suppose $f \in \mathcal{C}(\overline{\Omega} \times \partial\Omega)$ is such that*

(i) $f(z, w) = 0$ if and only if $z = w \in \partial\Omega$, and
(ii) there exist $\eta > 1$ and $\tau > 0$ such that

$$f(z, w) = a(z, w)\mathfrak{p}(z, w) + O\left(\mathfrak{p}(z, w)^\eta\right) \tag{1.3}$$

on $\Omega_\tau := \{(z, w) \in \overline{\Omega} \times \partial\Omega : ||z - w|| \le \tau\}$, where $\mathfrak{p}$ is the Levi polynomial of some strictly plurisubharmonic defining function of Ω (see Sect. 2) and a is some continuous non-vanishing function on Ω_τ.

Then, there exists a constant $l_{\mathrm{kor}} > 0$, independent of Ω, such that

$$\inf\{\mathrm{vol}(\Omega \setminus P) : P \in \mathcal{P}_n(f)\} \sim \frac{1}{2} l_{\mathrm{kor}} \left(\int_{\partial\Omega} \sigma_\Omega \right)^{\frac{3}{2}} \frac{1}{\sqrt{n}} \tag{1.4}$$

as $n \to \infty$.

In the tradition of div_{d-1} and ldiv_{d-1}, the constant l_{kor} above is named after Laguerre–Korányi tilings. Any such tiling comes from a collection of Korányi balls $\mathscr{K}$ covering $[0,1]^3$ in $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$ by minimizing the horizontal power functions $\text{hpow}(\cdot, K) : \mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \mathbb{R}$ associated with the balls K in $\mathscr{K}$ (see the Appendix for more details). If $\text{hcell}(K)$ denotes the tile associated with $K \in \mathscr{K}$, we obtain that

$$l_{\text{kor}} = \lim_{n\to\infty} \sqrt{n} \inf \left\{ - \sum_{K\in\mathscr{K}} \int_{\text{hcell}(K)} \text{hpow}(z', K) dz' : \#(\mathscr{K}) \leq n \right\}.$$

Such descriptions have been obtained for div_{d-1} and ldiv_{d-1} as well (see [10, 15] and [7]).

We believe that our proof of Theorem 1.1 can be generalized to higher dimensions, although the exposition becomes exceedingly complicated. We, therefore, merely state what we believe to be is the corresponding asymptotic formula when $\Omega \subset\subset \mathbb{C}^d$ and $f \in \mathcal{C}(\overline{\Omega} \times \partial\Omega)$ satisfy the hypothesis of the above theorem: there is a constant $c_d > 0$ such that

$$\inf\{\text{vol}(\Omega \setminus P) : P \in \mathcal{P}_n(f)\} \sim \frac{1}{2} c_d \left(\int_{\partial\Omega} \sigma_\Omega \right)^{\frac{d+1}{d}} \frac{1}{n^{1/d}} \tag{1.5}$$

as $n \to \infty$. Here, c_d is the d-dimensional version of l_{kor}. We encourage the reader to compare the exponents and decay rates in (1.5), (1.1) and (1.2). A common pattern emerges when we realize that the role played by the Euclidean metric on $\mathbb{R}^{d-1}$ in obtaining (1.1) and (1.2) is played by the Korányi metric on the $(2d-1)$-dimensional Heisenberg group in the case of (1.5). The former has Hausdorff dimension $d' = d-1$, while the latter has Hausdorff dimension $d' = 2d$. The exponent of the boundary measure and the power of $1/n$ in all three formulae now have the unified expressions $(d'+2)/d'$ and $2/d'$, respectively.

Let $\text{LP}(\Omega)$ denote the class of $f \in \mathcal{C}(\overline{\Omega} \times \partial\Omega)$ that satisfy conditions (i) and (ii) of Theorem 1.1. Then, $\text{LP}(\Omega)$ is invariant under biholomorphisms that extend ($\mathcal{C}^2$-)smoothly to the boundary. $\text{LP}(\Omega)$ is a natural class when working with strongly pseudoconvex domains and contains elements that yield analytic polyhedra. The Henkin–Ramirez generating function (see [16, §3] for details) is one such choice of f. So are $K_\Omega^{-1/(d+1)}$ and $S_\Omega^{-1/d}$, where K_Ω and S_Ω denote the Bergman kernel and Szegő kernel on $\Omega \subset \mathbb{C}^d$, respectively. In fact, these two choices of f are almost analytic extensions of any defining function of Ω. Since the Bergman kernel and almost analytic extensions of defining functions make sense in a context larger than that of strongly pseudoconvex domains, these provide potential candidates for f to obtain results like Theorem 1.1 in a more general setting. We support this fact with an example where the Fefferman hypersurface measure, though not defined everywhere, is zero almost everywhere with respect to the Hausdorff measure on the boundary. Let $\Omega = \mathbb{D}^2$ and $f(z, w) = (1 - z_1\overline{w_1})(1 - z_2\overline{w_2})$. Then, by choosing appropriate f-cuts with sources on the distinguished boundary, it can be shown that

$$\lim_{n\to\infty} \sqrt{n} \inf\{\text{vol}(\Omega \setminus P) : P \in \mathcal{P}_n(f)\} = 0$$

as $n \to \infty$. Note that f is a scalar multiple of $K_{\mathbb{D}^2}^{-1/2}$.

Organization of the paper Definitions, notation and terminology that feature in multiple sections are collected in Sect. 2. The proof of Theorem 1.1 is spread over subsequent sections. A critical lemma allows us to pass from LP(Ω) to a single representative—this lemma and other technical issues are dealt with in Sect. 3. In Sect. 4, we address the problem for certain model domains and model polyhedra. The rate of decay and the relevant exponents in (1.4) become evident in this section. We move from the model to the general case (locally), and from the local to the global case in Sects. 5 and 6, respectively. The Appendix contains a brief exposition on power diagrams in the Euclidean plane, and introduces a new tiling problem on the Heisenberg group. The latter emerged naturally in the course of this work, and seems indispensable in proving Theorem 1.1 (in particular, Lemma 6.1 from the Appendix is a crucial component of Lemma 5.6). The Appendix also contains bounds for l_{kor}.

2 Preliminaries

In this article, $\mathbb{N}_+$ denotes the set of all positive natural numbers. For $D \subseteq \mathbb{R}^n$, $\mathcal{C}(D)$ is the set of all continuous functions on D, and $\mathcal{C}^k(D)$, $k \geq 1$, denotes the set of all functions that are k-times continuously differentiable in some open neighborhood of D. If $A \subset B \subset \mathbb{R}^n$, $\text{int}_B A$ is the interior of A in the relative topology of B. The transpose of a vector v is denoted by v^{tr}. When well defined, $\text{J}_{\mathbb{R}} f(x)$ and $\text{Hess}_{\mathbb{R}} f(x)$ denote the real Jacobian and Hessian matrices, respectively, of f at x, $\text{J}_{\mathbb{C}} f(z)$ is the complex Jacobian matrix of f at z, and f^* denotes the pull-back operator induced by f on differential forms and measures. For brevity, we often abbreviate $\frac{\partial f}{\partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ to f_x and f_{xy}, respectively. In $\mathbb{C}^2$, we employ the notation

- $z = (z_1, z_2) = (x_1 + iy_1, x_2 + iy_2)$, $w = (w_1, w_2) = (u_1 + iv_1, u_2 + iv_2)$ for points;
- $\mathbb{B}_2(z; r)$ for the Euclidean ball centered at z and of radius r;
- $\left\langle \cdot, \cdot \right\rangle$ for the complex pairing between a co-vector and a vector;
- “ $'$ ” to indicate projection onto $\{y_2 = 0\} = \mathbb{C} \times \mathbb{R}$;
- A^{res} for $(A\big|_{\{y_2=0\}})' : \mathbb{C} \times \mathbb{R} \to \mathbb{C} \times \mathbb{R}$, where $A : \mathbb{C}^2 \to \mathbb{C}^2$;
- vol for the Lebesgue measure in $\mathbb{C}^2$;
- vol_3 for the Lebesgue measure in $\mathbb{C} \times \mathbb{R}$, and
- s for the standard Euclidean surface area measure on the boundary of a smooth domain.

In our analogy between convex and complex analysis, the role of convexity is played by pseudoconvexity:

Definition 2.1 A $\mathcal{C}^2$-smooth domain $\Omega \subset \mathbb{C}^d$ is called *strongly pseudoconvex* if it admits a defining function ρ in a neighborhood $U \supset \overline{\Omega}$ such that

$$\sum_{1\le j,k\le d} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z_k}}(z) v_j \overline{v_k} \ > 0 \quad \text{for } z \in \partial\Omega \text{ and } v = (v_1, \ldots, v_d) \in \mathbb{C}^d \setminus \{0\}$$

$$\text{satisfying } \sum_{j=1}^{d} \frac{\partial \rho}{\partial z_j}(z) v_j = 0. \tag{2.1}$$

A (possibly non-smooth) domain $\Omega \subset \mathbb{C}^d$ is called *pseudoconvex* if it can be exhausted by strongly pseudoconvex domains, i.e., $\Omega = \cup_{j\in\mathbb{R}} \Omega_j$ with each Ω_j strongly pseudoconvex and $\Omega_j \subseteq \Omega_k$ for $j < k$.

Remark We will heavily use the fact that any strongly pseudoconvex domain Ω admits a defining function ρ which is *strictly plurisubharmonic*, i.e., (2.1) holds for all $z \in U$ and $v \in \mathbb{C}^d \setminus \{0\}$.

We reintroduce the polyhedral objects of our study.

Definition 2.2 Let $\Omega \subset \mathbb{C}^2$ be a domain and $f \in \mathcal{C}(\overline{\Omega} \times \partial\Omega)$. Given a compact set $J \subset \partial\Omega$, an *f-polyhedron over J* is any set of the form

$$P = \{z \in \Omega : |f(z, w^j)| > \delta_j, \, j = 1, \ldots, n\}, \ (w^j, \delta_j) \in \partial\Omega \times (0, \infty),$$

such that $J \subset \partial\Omega \setminus \overline{P}$ and for every $j \in \{1, \ldots, n\}$, $|f(z, w^j)| < \delta_j$ for some $z \in J$. If Ω is bounded, then an f-polyhedron over $\partial\Omega$ is simply called an *f-polyhedron*. We call

- each (w^j, δ_j) a *source-size pair* of P;
- each $C(w^j, \delta_j; f) := \{z \in \overline{\Omega} : |f(z, w^j)| \le \delta_j\}$ a *cut* of P;
- each $F(w^j, \delta_j; f) := \{z \in \overline{\Omega} : |f(z, w^j)| = \delta_j, |f(z, w^l)| \ge \delta_l, \ l \ne j\}$ a *facet* of P;
- $(w^1, \ldots, w^n)$ and $(\delta_1, \ldots, \delta_n)$ the *source-tuple* and *size-tuple* of P, respectively.

We emphasize that, by definition, the cuts of an f-polyhedron over J cover J, and each of its cuts intersects J non-trivially.

Remarks When there is no ambiguity in the choice of f, we drop any reference to it from our notation for cuts and facets. Repetitions are permitted when listing the sources of an f-polyhedron. Thus, P—as in Definition 2.2—has at most n facets.

Let Ω, f, P and J be as in Definition 2.2 above. We will use the following notation.

- $\delta(P) := \max\{\delta_j : 1 \le j \le n \text{ and } (\delta_1, \ldots, \delta_n) \text{ is the size-tuple of } P\}$.
- $\mathcal{P}_n(f) :=$ the collection of all f-polyhedra in Ω with at most n facets.
- $\mathcal{P}_n(J; f) :=$ the collection of all f-polyhedra over J with at most n facets.
- $\mathcal{P}_n(J \subset H; f) := \{P \in \mathcal{P}_n(J; f) : \partial\Omega \setminus \overline{P} \subset H\}$, where $H \subset \partial\Omega$ is a compact superset of J.
- $v(\Omega; \mathscr{P}) := \inf\{\text{vol}(\Omega \setminus P) : P \in \mathscr{P}\}$, for any sub-collection $\mathscr{P} \subset \mathcal{P}_n(J \subset H; f)$.
- $v_n(f) := v(\Omega; \mathcal{P}_n(J \subset H; f))$, when the choice of Ω, J and H is unambiguous.

- $v_n(J \subset H) := v(\Omega; \mathcal{P}_n(J \subset H; f))$, when the choice of Ω and f is unambiguous.

We now introduce some terminology and notation that will be used repeatedly in Sect. 5.

- Let $\rho : U \to \mathbb{R}$ be $\mathcal{C}^2$-smooth. The Levi polynomial associated with ρ is the map $\mathfrak{p}_\rho : U \times U \to \mathbb{C}$ given by

$$\mathfrak{p}(z, w) = \sum_{j=1}^{2} \frac{\partial \rho}{\partial z_j}(w)(z_j - w_j) + \frac{1}{2} \sum_{j,k=1}^{2} \frac{\partial^2 \rho}{\partial z_j \partial z_k}(w)(z_j - w_j)(z_k - w_k).$$

  If the choice of ρ is unambiguous, we will use $\mathfrak{p}$ instead.
- Let $\rho : U \to \mathbb{R}$ be $\mathcal{C}^2$-smooth. The Cauchy–Leray map associated with ρ is the map $\mathfrak{l}_\rho : U \times U \to \mathbb{C}$ given by

$$\mathfrak{l}_\rho(z, w) = \sum_{j=1}^{2} \frac{\partial \rho}{\partial z_j}(w)(z_j - w_j).$$

- $\mathcal{S}_\lambda = \{(z_1, z_2) \in \mathbb{C}^2 : \rho^\lambda(z_1, z_2) < 0\}$, where $\rho^\lambda(z_1, z_2) = \lambda |z_1|^2 - \operatorname{Im} z_2$. When $\lambda = 1$, $\mathcal{S}_\lambda = \mathcal{S}$.
- For brevity, $\mathfrak{l}_\lambda := \mathfrak{l}_{\rho^\lambda}$, and $f_\lambda(z, w) := -2i\lambda \mathfrak{l}_\lambda(z, w)$ when $w \in \partial \mathcal{S}^\lambda$.
- As defined in Theorem 1.1, for any domain $\Omega \subset \mathbb{C}^2$ and $\tau > 0$, $\Omega_\tau := \{(z, w) \in \overline{\Omega} \times \partial\Omega : ||z - w|| < \tau\}$.

3 Some Technical Lemmas

Here, we restrict our attention to Jordan measurable domains $\Omega \subset \mathbb{C}^2$. J and H are compact subsets of $\partial\Omega$ such that $J \subset \operatorname{int}_{\partial\Omega} H$. We will concern ourselves with f-polyhedra over J that are constrained by H. We first prove a lemma that will allow us to work locally.

Lemma 3.1 *Let Ω, J and H be as above. Suppose there are $\delta_0 > 0$, $c > 0$ and $f, g \in \mathcal{C}(\overline{\Omega} \times H)$ such that*

(a) $\{z \in \overline{\Omega} : f(z, w) = 0\} = \{z \in \overline{\Omega} : g(z, w) = 0\} = \{w\}$, for any fixed $w \in H$,
(b) $C(w, \delta; f) \supseteq C(w, c\delta; g)$, for all $w \in H$ and $\delta < \delta_0$,
(c) $C(w, \delta; g)$ is Jordan measurable for each $w \in H$ and $\delta < c\delta_0$.

Then, for $P_n \in \mathcal{P}_n(J \subset H; f)$ such that $\lim_{n\to\infty} \operatorname{vol}(\Omega \setminus P_n) = 0$, *we have that* $\lim_{n\to\infty} \delta(P_n) = 0$.

Proof It suffices to show that for each $\delta < \delta_0$, there is a $b > 0$ such that $\operatorname{vol}(C(w, \delta; f)) > b$ for all $w \in H$. By condition (b), it is enough to show this for the cuts of g. By (a), $\operatorname{vol}\big(C(w, \delta; g)\big) > 0$ for each $w \in H$ and $\delta < c\delta_0$. Thus, it is enough to prove the continuity of $w \mapsto \operatorname{vol}\big(C(w, \delta; g)\big)$, $\delta < c\delta_0$, on the compact set H.

Fix a $\delta \in (0, c\delta_0)$. Let $\chi_w := \chi_{C(w,\delta;g)}$, where χ_A denotes the indicator function of A. For a given $w \in H$, consider a sequence of points $\{w^n\}_{n\in\mathbb{N}} \subset H$ that converges to w as $n \to \infty$. Then,

$$\lim_{n\to\infty} \chi_{w^n}(z) = \chi_w(z) \quad \text{for a.e. } z \in \overline{\Omega}. \tag{3.1}$$

To see this, consider a $z \in \overline{\Omega}$ such that $\chi_w(z) = 0$. Suppose there is a subsequence $\{w^{n_j}\}_{j\in\mathbb{N}} \subset \{w^n\}_{n\in\mathbb{N}}$ such that $\chi_{w^{n_j}}(z) = 1$. Then, $|g(z, w^{n_j})| \leq \delta$ but $\lim_{j\to\infty} |g(z, w^{n_j})| = |g(z, w)| \geq \delta$. This is only possible if $g(z, w) = \delta$. An analogous argument holds if $\chi_w(z) = 1$. Thus, $z \in \partial C(w, \delta; g)$. Due to assumption ($c$), this is a null set. Thus, (3.1) is true and we invoke Lebesgue's dominated convergence theorem to conclude that

$$\text{vol}\left(C(w^n, \delta; g)\right) = \int_{\overline{\Omega}} \chi_{w^n} d\omega \xrightarrow{n\to\infty} \int_{\overline{\Omega}} \chi_w d\omega = \text{vol}\left(C(w, \delta; g)\right),$$

where $\delta < c\delta_0$ and $\omega = \text{vol}$ is the Lebesgue measure on $\mathbb{C}^2$. □

Next, we prove a lemma that permits us to concentrate on a single representative of LP(Ω).

Lemma 3.2 *Let Ω, J and H be as above. Suppose $f, g \in \mathcal{C}(\overline{\Omega} \times H)$ are such that*

(i) $\{z \in \overline{\Omega} : f(z, w) = 0\} = \{z \in \overline{\Omega} : g(z, w) = 0\} = \{w\}$, *for any fixed $w \in H$, and*

(ii) there exist constants $\varepsilon \in (0, 1/3)$ and $\tau > 0$, such that

$$|f(z, w) - g(z, w)| \leq \varepsilon(|g(z, w) + |f(z, w)|) \tag{3.2}$$

on $\{(z, w) \in \overline{\Omega} \times H : ||z - w|| \leq \tau\}$.

Further, assume that the cuts of g are Jordan measurable and satisfy a doubling property as follows

(3.3) *there is a $\delta_g > 0$ and a continuous $D : [0, 16] \to \mathbb{R}$ so that, for any $n \in \mathbb{N}_+$, $(w^j, \delta_j) \in H \times (0, \delta_g)$, $1 \leq j \leq n$, and $t \in [0, 16]$,*

$$\text{vol}\left(\bigcup_{j=1}^{n} C(w^j, (1+t)\delta_j)\right) \leq D(t) \cdot \text{vol}\left(\bigcup_{j=1}^{n} C(w^j, \delta_j)\right). \tag{3.3}$$

Then, for every $\beta > 0$,

$$\limsup_{n\to\infty} n^\beta v_n(f) \leq D\left(\frac{(1+\varepsilon)^2}{(1-\varepsilon)^2} - 1\right) \limsup_{n\to\infty} n^\beta v_n(g); \tag{3.4}$$

$$\liminf_{n\to\infty} n^\beta v_n(f) \geq D\left(\frac{(1+\varepsilon)^4}{(1-\varepsilon)^4} - 1\right)^{-1} \liminf_{n\to\infty} n^\beta v_n(g), \tag{3.5}$$

where $v_n(h) = v\left(\Omega; \mathcal{P}_n(J \subset H; h)\right)$, $D_1(\varepsilon) =$ and $D_2(\varepsilon)$.

Proof Observe that if $\hat{\varepsilon} := \frac{1+\varepsilon}{1-\varepsilon}$, then inequality (3.2) may be transcribed as

$$|f(z,w)| \leq \hat{\varepsilon}|g(z,w)| \text{ and } |g(z,w)| \leq \hat{\varepsilon}|f(z,w)| \tag{3.6}$$

on $\{(z,w) \in \overline{\Omega} \times H : ||z-w|| \leq \tau\}$. Hence, for any $w \in H$ and $\delta > 0$,

$$C(w,\delta; f) \subseteq \mathbb{B}_2(w;\tau) \Rightarrow C(w,\delta; f) \subseteq C\left(w, \hat{\varepsilon}\delta; g\right); \tag{3.7}$$

$$C(w,\delta; g) \subseteq \mathbb{B}_2(w;\tau) \Rightarrow C(w,\delta; g) \subseteq C\left(w, \hat{\varepsilon}\delta; f\right). \tag{3.8}$$

We first show that

$$\limsup_{n\to\infty} n^{\beta} v_n(f) \leq D\left(\frac{(1+\varepsilon)^2}{(1-\varepsilon)^2} - 1\right) \limsup_{n\to\infty} n^{\beta} v_n(g).$$

Let $\xi > 1$. Assume that $L_{\sup} := \limsup_{n\to\infty} n^{\beta} v_n(g)$ is finite. Then, there is an $n_{\xi} \in \mathbb{N}_+$ such that for each $n \geq n_{\xi}$, we can pick a $Q_n \in \mathcal{P}_n(J \subset H; g)$ satisfying

$$\text{vol}(\Omega \setminus Q_n) \leq \xi L_{\sup} n^{-\beta}. \tag{3.9}$$

As the cuts of g are Jordan measurable, Lemma 3.1 implies that $\delta(Q_n) \to 0$ as $n \to \infty$. Consequently, n_{ξ} can be chosen so that (3.9) continues to hold, and for all source-size pairs (w,δ) of Q_n, $n \geq n_{\xi}$, we have that

(a) $\delta < \delta_g$ (see condition (3.3) on g);
(b) $C(w,\delta; g) \subset \mathbb{B}_2(w;\tau)$ and $C(w, 4\delta; g) \cap \partial\Omega \subset H$; and
(c) $C(w, 2\delta; f) \subset \mathbb{B}_2(w;\tau)$.

The second part of (b) is possible as each cut of Q_n is compelled to intersect J non-trivially, by definition. For a fixed source-size pair (w,δ) of Q_n, we have, due to (3.8) and (3.7),

$$C(w,\delta; g) \subseteq C(w, \hat{\varepsilon}\delta; f) \subseteq C\left(w, \hat{\varepsilon}^2\delta; g\right).$$

The second inclusion is valid as $\hat{\varepsilon}\delta \leq 2\delta$, thus permitting the use of (3.7), given (c).

We can now approximate Q_n by an f-polyhedron by setting

$$\widetilde{Q_n} := \{z \in \Omega : |g(z,w)| > \hat{\varepsilon}^2\delta, (w,\delta) \text{ is a source-size pair of } Q_n\};$$

$$P_n := \{z \in \Omega : |f(z,w)| > \hat{\varepsilon}\delta, (w,\delta) \text{ is a source-size pair of } Q_n\}.$$

Our assumptions imply that $\widetilde{Q_n}$ and P_n are in $\mathcal{P}_n(J \subset H; g)$ and $\mathcal{P}_n(J \subset H; f)$, respectively. From the above inclusions, we have that $\widetilde{Q_n} \subseteq P_n \subseteq Q_n$, $n \geq n_{\xi}$. Hence, by property (3.3) of g and (3.9), we see that

$$\begin{aligned} n^{\beta} v_n(f) \leq n^{\beta}\, \text{vol}(\Omega \setminus P_n) &\leq n^{\beta}\, \text{vol}\left(\Omega \setminus \widetilde{Q_n}\right) \\ &\leq D\left(\hat{\varepsilon}^2 - 1\right) n^{\beta}\, \text{vol}(\Omega \setminus Q_n) \\ &\leq \xi D\left(\hat{\varepsilon}^2 - 1\right) L_{\sup}, \end{aligned}$$

for $n \geq n_{\xi}$. As $\xi > 0$ was arbitrary and $\hat{\varepsilon} = \frac{1+\varepsilon}{1-\varepsilon}$, (3.4) follows.

To complete this proof, we show that

$$\liminf_{n\to\infty} n^\beta v_n(f) \geq D\left(\frac{(1+\varepsilon)^4}{(1-\varepsilon)^4}-1\right)^{-1} \liminf_{n\to\infty} n^\beta v_n(g).$$

For this, fix a $\xi > 1$, and assume that $L_{\inf} := \liminf_{n\to\infty} n^\beta v_n(g)$ is finite. Thus, there is an $n_\xi \in \mathbb{N}_+$ such that

$$v_n(g) \geq \frac{1}{\xi} L_{\inf} n^{-\beta}; \text{ for } n \geq n_\xi. \tag{3.10}$$

For each n, we pick an $R_n \in \mathcal{P}_n(J \subset H; f)$ that satisfies

$$v(\Omega \setminus R_n) \leq \xi v_n(f). \tag{3.11}$$

Now, we may also assume that $\liminf_{n\to\infty} n^\beta v_n(f) < \infty$ (else, there is nothing to prove), thus obtaining that $v_n(f) \to 0$ for infinitely many $n \in \mathbb{N}_+$. But, as $v_n(f)$ is decreasing in n, we get that $v_n(f) \to 0$ for all $n \in \mathbb{N}_+$. Now, due to (3.8), it is possible to choose δ small enough so that

$$C\left(w, \frac{\delta}{\hat{\varepsilon}}; g\right) \subseteq C(w, \delta; f),$$

for each $w \in H$. As the cuts of g are Jordan measurable (there is no such assumption on the cuts of f), we invoke Lemma 3.1 to conclude that $\delta(R_n) \to 0$ as $n \to \infty$. As before, we find a new n_ξ such that (3.10) continues to hold, and for all $n \geq n_\xi$ and all source-size pairs (w, δ) of R_n, we have

- (a') $\delta < \delta_g$ (see condition (3.3) on g);
- (b') $C(w, 4\delta; f) \subset \mathbb{B}_2(w; \tau)$ and $C(w, 4\delta; f) \cap \partial\Omega \subset H$; and
- (c') $C(w, 2\delta; g) \subset \mathbb{B}_2(w; \tau)$.

Then, as before

$$C\left(w, \frac{\delta}{\hat{\varepsilon}}; g\right) \subseteq C(w, \delta; f) \subseteq C\left(w, \hat{\varepsilon}\delta; g\right) \subseteq C\left(w, \hat{\varepsilon}^2\delta; f\right) \subseteq C\left(w, \hat{\varepsilon}^3\delta; g\right). \tag{3.12}$$

We now approximate R_n with an n-faceted g-polyhedron, using

$$\widetilde{R_n} := \left\{z \in \Omega : |f(z, w)| > \hat{\varepsilon}^2\delta, (w, \delta) \text{ is a source-size pair of } R_n\right\};$$
$$S_n := \left\{z \in \Omega : |g(z, w)| > \hat{\varepsilon}\delta, (w, \delta) \text{ is a source-size pair of } R_n\right\}.$$

Our assumptions are designed to ensure that $\widetilde{R_n} \in \mathcal{P}_n(J \subset H; f)$ and $S_n \in \mathcal{P}_n(J \subset H; g)$. From the above inclusions, we have that

$$\widetilde{R_n} \subseteq S_n \subseteq R_n, \; n \geq n_\xi.$$

Moreover, the first and last inclusions in (3.12) and the assumption (3.3) on g (note that $\hat{\varepsilon}^4 < 16$) imply that

$$\begin{aligned}\operatorname{vol}(\Omega \setminus \widetilde{R_n}) - \operatorname{vol}(\Omega \setminus R_n) &\leq \operatorname{vol}\left(\bigcup_{(w,\delta)\in\Lambda_n} C\left(w, \hat{\varepsilon}^3\delta; g\right) - \bigcup_{(w,\delta)\in\Lambda_n} C\left(w, \frac{\delta}{\hat{\varepsilon}}; g\right)\right) \\ &\leq D\big(\hat{\varepsilon}^4 - 1\big) \operatorname{vol}(\Omega \setminus R_n), \end{aligned} \tag{3.13}$$

where Λ_n is the set of source-size pairs of R_n.

Therefore, using (3.13) and (3.11), we see that

$$\begin{aligned}\frac{1}{\xi} L_{\inf} n^{-\beta} \leq v_n(g) \leq \operatorname{vol}(\Omega \setminus S_n) &\leq \operatorname{vol}\left(\Omega \setminus \widetilde{R_n}\right) \\ &\leq D\big(\hat{\varepsilon}^4 - 1\big) \operatorname{vol}(\Omega \setminus R_n) \\ &\leq D\big(\hat{\varepsilon}^4 - 1\big) \xi v_n(f).\end{aligned}$$

Therefore,

$$n^{\beta} v_n(f) \geq \xi^{-2} D\big(\hat{\varepsilon}^4 - 1\big)^{-1} L_{\inf}, \quad n \geq n_{\xi}.$$

As $\xi > 0$ was arbitrary and $\hat{\varepsilon} = \frac{1+\varepsilon}{1-\varepsilon}$, (3.5) follows. □

Remark 3.3 In practice, f and g may only be defined on $(\overline{\Omega} \cap U) \times H$ for some open set $U \subset \mathbb{C}^2$ containing a τ-neighborhood of H, while satisfying the analogous version of condition (i) there. As the remaining hypothesis (and indeed the result itself) depends only on the values of f and g on an arbitrarily thin tubular neighborhood of H in $\overline{\Omega}$, we may replace f (and, similarly, g) by f_{e} to invoke Lemma 3.2, where

$$f_{\mathsf{e}} := f(z, w)\varsigma(||z - w||^2) + ||z - w||^2(1 - \varsigma(||z - w||^2))$$

for some non-negative $\varsigma \in \mathcal{C}^{\infty}(\mathbb{R})$ such that $\varsigma(x) = 1$ when $x \leq \tau^2/2$ and $\varsigma(x) = 0$ when $x \geq \tau^2$. We will do so without comment, when necessary.

4 Approximating Model Domains

As a first step, we examine volume approximations of the Siegel domain by a particular class of analytic polyhedra. This problem enjoys a connection with Laguerre-type tilings of the Heisenberg surface equipped with the Korányi metric (see the Appendix for further details).

Let $\mathcal{S} := \{(z_1, x_2 + iy_2) \in \mathbb{C}^2 : y_2 > |z_1|^2\}$ and $f_{\mathcal{S}}(z, w) = z_2 - \overline{w_2} - 2iz_1\overline{w_1}$. We view $\mathbb{C} \times \mathbb{R}$ as the first Heisenberg group, $\mathbb{H}$, with group law

$$(z_1, x_2) \cdot_{\mathbb{H}} (w_1, u_2) = (z_1 + w_1, x_2 + u_2 + 2\operatorname{Im}(z_1\overline{w}_1))$$

and the left-invariant Korányi gauge metric (see [8, Sect. 2.2])

$$d_{\mathbb{H}}((z_1, x_2), (w_1, u_2)) := ||(w_1, u_2)^{-1} \cdot_{\mathbb{H}} (z_1, x_2)||_{\mathbb{H}},$$

where $||(z_1, x_2)||_{\mathbb{H}}^4 := |z_1|^4 + x_2^2$. Observe that, for any cut $C(w, \delta) = C(w, \delta; f_{\mathcal{S}})$, $w \in \partial\mathcal{S}$, $C(w, \delta)'$ is the set

$$K(w', \sqrt{\delta}) = \{(z_1, x_2) \in \mathbb{C}\times\mathbb{R} : |z_1 - w_1|^4 + (x_2 - u_2 + 2\,\mathrm{Im}(z_1\overline{w_1}))^2 \leq \delta^2\}, \quad (4.1)$$

which is the ball of radius $\sqrt{\delta}$ centered at w', in the Korányi metric.

Notation We will use the following notation in this section:

- $I^r := \{(x_1 + iy_1, x_2) \in \mathbb{C} \times \mathbb{R} : 0 \leq x_1 \leq r,\ 0 \leq y_1 \leq r,\ 0 \leq x_2 \leq r^2\}, r > 0$.
- $\hat{I}^r := I^{2r} - \left(\frac{r}{2} + i\frac{r}{2}, \frac{3r^2}{2}\right), r > 0$. $I^r \subset \hat{I}^r$ and they are concentric.
- $v_n(J \subset H) := v(\mathcal{S}; \mathcal{P}_n(J \subset H; f_{\mathcal{S}}))$, for $J \subset H \subset \partial\mathcal{S}$. If $J \subset H \subset \mathbb{C} \times \mathbb{R}$, $v_n(J \subset H)$ is meaningful in view of the obvious correspondence between $\mathbb{C} \times \mathbb{R}$ and $\partial\mathcal{S}$.

Lemma 4.1 *Let $I = I^1$ and $\hat{I} = \hat{I}^1$. There exists a positive constant $l_{\mathrm{kor}} > 0$ such that*

$$v_n(I \subset \hat{I}) \sim \frac{l_{\mathrm{kor}}}{\sqrt{n}}$$

as $n \to \infty$.

Proof Simple calculations show that

$$\mathrm{vol}(C(w, \delta)) = \frac{2\pi}{3}\delta^3 \quad (4.2)$$

$$\mathrm{vol}_3(K(w', \sqrt{\delta})) = \frac{\pi^2}{2}\delta^2 \quad (4.3)$$

for all $w \in \partial\mathcal{S}$ and $\delta > 0$.

We utilize a special tiling in $\mathbb{C} \times \mathbb{R}$. Let $k \in \mathbb{N}_+$ and consider the following points in $\mathbb{C} \times \mathbb{R}$:

$$v_{pqr} := \left(\frac{p}{k} + i\frac{q}{k}, \frac{r}{k^2}\right),\ (p, q, r) \in \Sigma_k,$$

where $\Sigma_k := \left\{(p, q, r) \in \mathbb{Z}^3 : -2q \leq r \leq k^2 - 1 + 2p,\ 0 \leq p, q \leq k - 1\right\}$. Observe that $\mathrm{card}(\Sigma_k) = k^4 + 2k^3 - 2k^2$. Now, we set $E_{pqr} := v_{pqr} \cdot_{\mathbb{H}} I^{\frac{1}{k}}$, and note that $I \subset \cup_{\Sigma_k} E_{pqr} \subset \hat{I}$, for all $k \in \mathbb{N}_+$. (See Fig. 1.)

1. We first show that there is a constant $\alpha_1 > 0$ such that

$$v_n(I \subset \hat{I}) \leq \frac{\alpha_1}{\sqrt{n}} \quad (4.4)$$

for all $n \in \mathbb{N}_+$.

For this, let

$$u_{pqr} := \text{center of } E_{pqr} = v_{pqr} \cdot_{\mathbb{H}} \left(\frac{1}{2k} + i\frac{1}{2k}, \frac{1}{2k^2}\right),\ (p, q, r) \in \Sigma_k,\ k \in \mathbb{N}_+.$$

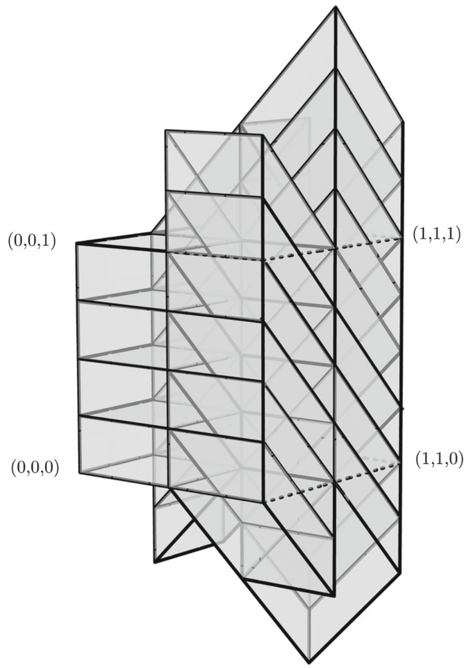

Fig. 1 The 24 tiles E_{pqr} when $k = 2$

Then, the Korányi ball $K\left(u_{pqr}, \frac{\sqrt[4]{5}}{\sqrt[4]{2}k}\right)$ (see (4.1)) contains E_{pqr} and is contained in $\hat{I}$. Hence, if $w_{pqr} \in \partial\mathcal{S}$ is such that $w_{pqr}{}' = u_{pqr}$, the cuts

$$C\left(w_{pqr}, \frac{\sqrt{5}}{\sqrt{2}k^2}; f_{\mathcal{S}}\right), \ (p, q, r) \in \Sigma_k,$$

define P_k, an $f_{\mathcal{S}}$-polyhedron with $k^4 + 2k^3 - 2k^2$ facets. In fact, $P_k \in \mathcal{P}_{k^4+2k^3-2k^2}(I \subset \hat{I};\ f_{\mathcal{S}})$, for all $k \in \mathbb{N}_+$, where we identify I and $\hat{I}$ with their images in ∂S under the map $(z_1, x_2) \mapsto (z_1, x_2 + i|z_1|^2)$. Therefore, using (4.2)

$$\begin{aligned} v_{k^4+2k^3-2k^2}(I \subset \hat{I}) &\leq \mathrm{vol}(\mathcal{S} \setminus P_k) \\ &\leq \mathrm{vol}\left(\bigcup_{\Sigma_k} C\left(w_{pqr}, \frac{\sqrt{5}}{\sqrt{2}k^2}\right)\right) \\ &\leq \frac{2\pi}{3}\left(\frac{\sqrt{5}}{\sqrt{2}k^2}\right)^3 (k^4 + 2k^3 - 2k^2) = \frac{5\sqrt{5}\pi}{3\sqrt{2}} \frac{(k^4 + 2k^3 - 2k^2)}{k^6}, \end{aligned}$$

$k \in \mathbb{N}_+$. Now, for a given $n \in \mathbb{N}_+$, choose k such that $k^4 + 2k^3 - 2k^2 \leq n \leq (k+1)^4 + 2(k+1)^3 - 2(k+1)^2$. Then, one can easily find a $\alpha_1 > 0$ such that

$$\begin{aligned} v_n(I \subset \hat{I})\sqrt{n} &\leq v_{k^4+2k^3-2k^2}(I \subset \hat{I})\sqrt{(k+1)^4+2(k+1)^3-2(k+1)^2} \\ &\leq \frac{5\sqrt{5}\pi}{3\sqrt{2}} \frac{(k^4+2k^3-2k^2)\sqrt{(k+1)^4+2(k+1)^3-2(k+1)^2}}{k^6} \\ &\leq \alpha_1. \end{aligned}$$

2. Next, we show that there is an $\alpha_2 > 0$ such that

$$v_n(I \subset \hat{I}) \geq \frac{\alpha_2}{\sqrt{n}} \tag{4.5}$$

for $n \in \mathbb{N}_+$.

If finitely many Korányi balls of radii $\sqrt{\rho_1}, \ldots, \sqrt{\rho_k}$ cover I, then (4.3) yields

$$(\sqrt{\rho}_1)^4 + \cdots + (\sqrt{\rho}_k)^4 \geq \frac{2}{\pi^2}\,\mathrm{vol}_3(I) = \frac{2}{\pi^2}. \tag{4.6}$$

We will also need the following mean inequality (a consequence of Jensen's inequality)

$$\left(\frac{\rho_1^{d+1}+\cdots+\rho_k^{d+1}}{k}\right)^{\frac{1}{d+1}} \geq \left(\frac{\rho_1^{d-1}+\cdots+\rho_k^{d-1}}{k}\right)^{\frac{1}{d-1}}, \tag{4.7}$$

for positive ρ_j, $1 \leq j \leq k$, and $d > 1$.

Now, fix a $\xi > 1$. Let $P_n \in \mathcal{P}_n(I \subset \hat{I};\, f_{\mathcal{S}})$ be such that

$$\mathrm{vol}(\mathcal{S} \setminus P_n) \leq \xi v_n(I \subset \hat{I}). \tag{4.8}$$

Let C_j and K_j, $j = 1, \ldots, n$, be the cuts and their projections, respectively, of P_n. Now, $\mathcal{K}_n := \{K_j, j = 1, \ldots, n\}$ is a finite covering of I, so by the Wiener covering lemma (see [13, Lemma 4.1.1] for a proof that generalizes to metric spaces), we can find, on renumbering the indices, disjoint Korányi balls $K_1, \ldots, K_k \in \mathcal{K}_n$ of radii $\sqrt{\rho_1}, \ldots, \sqrt{\rho_k}$, such that $\cup_{K \in \mathcal{K}_n} K \subset \cup_{1 \leq j \leq k} 3K_j$, where, for $j = 1, \ldots, k$, $3K_j$ has the same center as K_j but thrice its radius. Let C_j denote the cut that projects to K_j, $j = 1, \ldots, k$. It follows from (4.8), (4.2) and the inequalities (4.7) (for $d = 5$) and (4.6) that

$$\begin{aligned} v_n(I \subset \hat{I})\sqrt{n} &\geq \frac{1}{\xi}\,\mathrm{vol}\left(\bigcup_{j=1}^{k} C_j\right)\sqrt{k} \\ &= \frac{1}{\xi}\left(\sum_{i=1}^{k}\mathrm{vol}(C_j)\right)\sqrt{k} = \frac{2\pi}{3\xi}\left(\rho_1^3+\cdots+\rho_k^3\right)\sqrt{k} \\ &= \frac{2\pi}{3^7\xi}\left((9\rho_1)^3+\cdots+(9\rho_k)^3\right)\sqrt{k} = \frac{2\pi}{3^7\xi}\left((3\sqrt{\rho_1})^6+\cdots+(3\sqrt{\rho_k})^6\right)k^{\frac{2}{4}} \end{aligned}$$

$$\geq \frac{2\pi}{3^7\xi}\left((3\sqrt{\rho_1})^4+\cdots+(3\sqrt{\rho_k})^4\right)^{\frac{6}{4}}$$
$$\geq \frac{4\sqrt{2}}{\pi^2 3^7\xi}\operatorname{vol}_3(I)^{\frac{3}{2}} = \frac{4\sqrt{2}}{\pi^2 3^7\xi} > 0,\ \text{for}\, n = n_0, n_0+1, \ldots$$

As $\xi > 1$ was arbitrary, we have proved (4.5).

3. Define

$$l_{\text{kor}} = \liminf_{n\to\infty} v_n(I \subset \hat{I})\sqrt{n}.$$

By (4.5) and (4.4), $0 < l_{\text{kor}} < \infty$. We now show that

$$l_{\text{kor}} = \lim_{n\to\infty} v_n(I \subset \hat{I})\sqrt{n}. \tag{4.9}$$

For this, it suffices to show that for every $\xi > 1$, if $n_0 \in \mathbb{N}_+$ is chosen so that

$$v_{n_0}(I \subset \hat{I})\sqrt{n_0} \leq \xi l_{\text{kor}} \tag{4.10}$$

then,

$$v_n(I \subset \hat{I})\sqrt{n} \leq \xi^4 l_{\text{kor}} \tag{4.11}$$

for n sufficiently large.

Now, let $P_{n_0} \in \mathcal{P}_{n_0}(I \subset \hat{I};\, f_{\mathcal{S}})$ be such that

$$\operatorname{vol}(\mathcal{S} \setminus P_{n_0}) \leq \xi v_{n_0}(I \subset \hat{I}).$$

For any $w \in \partial\mathcal{S}$ and $k \in \mathbb{N}_+$, let $A_{w,k} : \mathbb{C}^2 \to \mathbb{C}^2$ be the biholomorphism

$$(z_1, z_2) \mapsto \left(w_1 + \frac{1}{k}z_1,\, w_2 + \frac{1}{k^2}z_2 - \frac{2i}{k}z_1\overline{w_1}\right).$$

Then, $A_{w,k}$ has the following properties:

- $A^{\text{res}}_{w,k}(z') = w' \cdot_{\mathbb{H}} \left(\frac{1}{k}z_1,\, \frac{1}{k^2}x_2\right)$;
- $A_{w,k}(\mathcal{S}) = \mathcal{S}$;
- $A_{w,k}(P_{n_0}) \in \mathcal{P}_{n_0}(w' \cdot_{\mathbb{H}} I^{\frac{1}{k}} \subset w' \cdot_{\mathbb{H}} \hat{I}^{\frac{1}{k}};\, f_{\mathcal{S}})$; and
- $\operatorname{vol}(\mathcal{S} \setminus A_{w,k}(P_{n_0})) \leq \xi \frac{v_{n_0}(I\subset\hat{I})}{k^6}$.

As a consequence,

$$P := \bigcup_{(p,q,r)\in\Sigma_k} A_{v_{pqr},k}(P_{n_0})$$

satisfies the following conditions:

- $P \in \mathcal{P}_{n_0(k^4+2k^3-2k^2)}(I \subset \hat{I};\, f_{\mathcal{S}})$
- $\operatorname{vol}(\mathcal{S} \setminus P) \leq \xi v_{n_0}(I \subset \hat{I})\frac{k^4+2k^3-2k^2}{k^6}$.

Hence, by assumption (4.10),

$$v_{n_0(k^4+2k^3-2k^2)}(I \subset \hat{I})\sqrt{n_0(k^4+2k^3-2k^2)} \leq \xi v_{n_0}(I \subset \hat{I})\sqrt{n_0}\frac{(k^4+2k^3-2k^2)^{\frac{3}{2}}}{k^6}$$
$$\leq \xi^2 v_{n_0}(I \subset \hat{I})\sqrt{n_0} \leq \xi^3 l_{\text{kor}}, \quad (4.12)$$

for sufficiently large k. Choose k_0 so that (4.12) holds and $\frac{(k+1)^4+2(k+1)^3-2(k+1)^2}{k^4+2k^3-2k^2} \leq \xi^2$ for $k > k_0$. For $n \geq n_0(k_0^4+2k_0^3-2k_0^2)$, let k be such that $n_0(k^4+2k^3-2k^2) \leq n \leq n_0((k+1)^4+2(k+1)^3-2(k+1)^2)$. Consequently,

$$v_n(I \subset \hat{I})\sqrt{n} \leq v_{n_0(k^4+2k^3-2k^2)}(I \subset \hat{I})\sqrt{n_0((k+1)^4+2(k+1)^3-2(k+1)^2)}$$
$$\leq \xi^3 l_{\text{kor}}\sqrt{\frac{(k+1)^4+2(k+1)^3-2(k+1)^2}{k^4+2k^3-2k^2}} \leq \xi^4 l_{\text{kor}},$$

by (4.12). We have proved (4.11) and, therefore, our claim (4.9). □

Our choice of the unit square in the above lemma facilitates the computation for polyhedra lying above more general Jordan measurable sets in the boundary of $\mathcal{S}$.

Lemma 4.2 *Let $J, H \subset \partial\mathcal{S}$ be compact and Jordan measurable with $J \subset \text{int}_{\partial\mathcal{S}}H$. Then*

$$v_n(J \subset H) \sim \text{vol}_3(J')^{\frac{3}{2}} l_{\text{kor}}\frac{1}{\sqrt{n}}$$

as $n \to \infty$.

Proof 1. We first show that

$$\limsup_{n\to\infty} v_n(J \subset H)\sqrt{n} \leq l_{\text{kor}} \text{vol}_3(J')^{\frac{3}{2}}. \quad (4.13)$$

Let $\xi > 1$ be fixed. As J is Jordan measurable, we can find m points $v^1, \ldots, v^m \in \mathbb{C} \times \mathbb{R}$ and some $r > 0$, such that

$$J' \subset \bigcup_1^m \left(v^j \cdot_{\mathbb{H}} I^r\right) \subset \bigcup_1^m \left(v^j \cdot_{\mathbb{H}} \hat{I}^r\right) \subset H' \quad (4.14)$$

and

$$m\, \text{vol}_3(I^r) \leq \xi\, \text{vol}_3(J'). \quad (4.15)$$

Now, observe that

$$\sqrt{k}\,\frac{v_k(v^j \cdot_{\mathbb{H}} I^r \subset v^j \cdot_{\mathbb{H}} \hat{I}^r)}{\text{vol}_3(I^r)^{\frac{3}{2}}} = \sqrt{k}\,\frac{v_k(I^r \subset \hat{I}^r)}{\text{vol}_3(I^r)^{\frac{3}{2}}} = \sqrt{k}\, v_k(I \subset \hat{I}). \quad (4.16)$$

Thus, due to (4.14), Lemma 4.1, (4.16) and (4.15), we have

$$\begin{aligned} v_{km}(J \subset H)\sqrt{km} &\le \sum_{j=1}^{m} v_k(v^j \cdot_{\mathbb{H}} I^r \subset v^j \cdot_{\mathbb{H}} \hat{I}^r)\sqrt{k}\sqrt{m} \\ &\le \xi l_{\mathrm{kor}} \operatorname{vol}_3(I^r)^{\frac{3}{2}} m^{\frac{3}{2}} \\ &\le \xi^{\frac{5}{2}} l_{\mathrm{kor}} \operatorname{vol}_3(J')^{\frac{3}{2}} \end{aligned} \tag{4.17}$$

for k sufficiently large. Choose $k_0 \in \mathbb{N}_+$ such that for $k \ge k_0$, (4.17) holds and $\sqrt{(k+1)/k} \le \xi$. For sufficiently large n, we can find a $k \ge k_0$ such that $mk \le n \le m(k+1)$. Hence,

$$\begin{aligned} v_n(J \subset H)\sqrt{n} &\le v_{km}(J \subset H)\sqrt{(k+1)m} \\ &\le \xi^{\frac{5}{2}} l_{\mathrm{kor}} \operatorname{vol}_3(J')^{\frac{3}{2}} \sqrt{\frac{k+1}{k}} \\ &\le \xi^{\frac{7}{2}} l_{\mathrm{kor}} \operatorname{vol}_3(J')^{\frac{3}{2}}. \end{aligned}$$

As $\xi > 1$ was arbitrarily fixed, we have proved (4.13).

2. It remains to show that

$$\liminf_{n\to\infty} v_n(J \subset H)\sqrt{n} \ge l_{\mathrm{kor}} \operatorname{vol}_3(J')^{\frac{3}{2}}. \tag{4.18}$$

Once again, fix a $\xi > 1$. The Jordan measurability of J ensures that there are pairwise disjoint sets $I_1, \dots, I_m$, where $I_j = v^j \cdot_{\mathbb{H}} I^{r_j}$ for some $r_j > 0$ and $v^j \in \mathbb{C}\times\mathbb{R}$, $1 \le j \le m$, such that

$$\bigcup_1^m I_j \subset J' \text{ and } \bigcup_1^m \hat{I}_j \subset J', \tag{4.19}$$

where $\hat{I}_j = v^j \cdot_{\mathbb{H}} \hat{I}^{r_j}$, and

$$\operatorname{vol}_3(J') \le \xi \sum_{j=1}^{m} \operatorname{vol}_3(I_j). \tag{4.20}$$

Choose a $P_n \in \mathcal{P}_n(J \subset H; f_{\mathcal{S}})$ such that $v(\mathcal{S} \setminus P_n) \le \xi v_n(J \subset H)$ and let n_j denote the number of cuts of P_n whose projections intersect I_j and are contained in $\hat{I}_j$. By part 1, $v_n(J \subset H) \to 0$ as $n \to \infty$. Thus, recalling (4.2), $\delta(P_n) \to 0$ as $n \to \infty$. So, we may choose n so large that the projections of these n_j cuts, in fact, cover I_j and no two cuts of P whose projections intersect two different I_j's intersect. Therefore,

$$n_1 + \cdots + n_m \le n. \tag{4.21}$$

By Lemma 4.1 and (4.16), there is an $n_0 \in \mathbb{N}_+$ such that

$$v_k(I_j \subset \hat{I}_j) \geq \frac{1}{\xi} l_{\text{kor}} \operatorname{vol}_3(I_j)^{\frac{3}{2}} \frac{1}{\sqrt{k}} \tag{4.22}$$

for $k \geq n_0$ and $j = 1, \ldots, m$. We may further increase n to ensure that

$$n_j \geq n_0 \text{ for } j = 1, \ldots, m.$$

Consequently, by (4.19) and (4.22), we have,

$$v_n(J \subset H) \geq \frac{1}{\xi} \sum_{j=1}^{m} v_{n_j}(I_j \subset \hat{I}_j) \geq \frac{l_{\text{kor}}}{\xi^2} \sum_{j=1}^{m} \frac{\operatorname{vol}_3(I_j)^{\frac{3}{2}}}{\sqrt{n_j}}.$$

Now, Hölder's inequality yields

$$\sum_{j=1}^{m} \operatorname{vol}_3(I_j) = \sum_{j=1}^{m} \left(\frac{\operatorname{vol}_3(I_j)}{n_j^{1/3}} \right) n_j^{1/3} \leq \left(\sum_{j=1}^{m} \frac{\operatorname{vol}_3(I_j)^{3/2}}{n_j^{1/2}} \right)^{\frac{2}{3}} \left(\sum_{j=1}^{m} n_j \right)^{\frac{1}{3}}.$$

Using this, (4.20) and (4.21), we obtain

$$v_n(J \subset H) \geq \frac{l_{\text{kor}}}{\xi^2} \left(\sum_{j=1}^{m} \operatorname{vol}_3(I_j) \right)^{\frac{3}{2}} \left(\frac{1}{\sum_1^m n_j} \right)^{\frac{1}{2}} \geq \frac{l_{\text{kor}}}{\xi^{7/2}} \operatorname{vol}_3(J')^{\frac{3}{2}} \frac{1}{\sqrt{n}}$$

for n sufficiently large. As the choice of $\xi > 1$ was arbitrary, (4.18) now stands proved. □

As a final remark, we extend the above lemma to a class of slightly more general model domains in order to illustrate the effect of the Levi determinant on our asymptotic formula.

Corollary 4.3 *Let $\mathcal{S}_\lambda := \{(z_1, x_2 + iy_2) \in \mathbb{C}^2 : y_2 > \lambda |z_1|^2\}$ and $f_{\mathcal{S}_\lambda}(z, w) = \lambda(z_2 - \overline{w_2}) - 2i\lambda^2(z_1 \overline{w_1})$, for $\lambda > 0$. Let $J, H \subset \partial\mathcal{S}_\lambda$ be compact and Jordan measurable with $J \subset \mathrm{int}_{\partial\mathcal{S}_\lambda} H$. Then*

$$v_n(\mathcal{S}_\lambda; J \subset H) := v(\mathcal{S}_\lambda; \mathcal{P}_n(J \subset H; f_{\mathcal{S}_\lambda})) \sim \lambda^{\frac{1}{2}} \operatorname{vol}_3(J')^{\frac{3}{2}} l_{\text{kor}} \frac{1}{\sqrt{n}}$$

as $n \to \infty$.

Proof Let $\Xi : \mathbb{C}^2 \to \mathbb{C}^2$ be the biholomorphism $\Xi : (z_1, z_2) \mapsto (\lambda z_1, \lambda z_2)$. Then, $\mathcal{S} = \Xi(\mathcal{S}_\lambda)$ and $f_{\mathcal{S}_\lambda}(z, w) = f_{\mathcal{S}}(\Xi(z), \Xi(w))$. Therefore, there is a bijective correspondence between $\mathcal{P}_n(J \subset H; f_{\mathcal{S}_\lambda})$ and $\mathcal{P}_n(\Xi J \subset \Xi H; f_{\mathcal{S}})$ given by $P \mapsto \Xi P$.

Now, as $\det(\mathrm{J}_{\mathbb{R}}\,\Xi) \equiv \lambda^4$ and $\det(\mathrm{J}_{\mathbb{R}}\,\Xi^{\mathrm{res}}) \equiv \lambda^3$, we have

$$\frac{v_n(\mathcal{S}_\lambda;\, J \subset H)}{\mathrm{vol}_3(J')^{\frac{3}{2}}} = \frac{\lambda^{-4} v_n(\mathcal{S};\, \Xi J \subset \Xi H)}{\lambda^{-\frac{9}{2}}\,\mathrm{vol}_3(\Xi^{\mathrm{res}} J')^{\frac{3}{2}}} \sim \lambda^{\frac{1}{2}} l_{\mathrm{kor}} \frac{1}{\sqrt{n}}.$$

□

5 Local Estimates Via Model Domains

Lemma 3.2 suggests a way to locally compare the volume-minimizing approximations drawn from two different classes of f-polyhedra which exhibit some comparability. In this section, our main goal is Lemma 5.5 where we set up a local correspondence between Ω and a model domain $\mathcal{S}_\lambda$, pull back the special cuts given by $f_{\mathcal{S}_\lambda}$ (defined in Sect. 4) via this correspondence, and establish a (3.2)-type relationship between the pulled-back cuts and those coming from the Levi polynomial of a defining function of Ω. First, we note a useful estimate on the Levi polynomial.

Lemma 5.1 *Let Ω be a $\mathcal{C}^2$-smooth strongly pseudoconvex domain. Suppose $\rho \in \mathcal{C}^2(\mathbb{C}^2)$ is a strictly plurisubharmonic defining function of Ω. Then, there exist constants $c > 0$ and $\tau > 0$ such that*

$$||z - w||^2 \leq c|\mathfrak{p}(z, w)|, \tag{5.1}$$

on $\Omega_\tau = \{(z, w) \in \overline{\Omega} \times \partial\Omega : ||z - w|| \leq \tau\}$, where $\mathfrak{p}$ is the Levi polynomial of ρ.

Proof The second-order Taylor expansion of ρ about $w \in \partial\Omega$ gives:

$$-2\,\mathrm{Re}\,\mathfrak{p}(z, w) = -\rho(z) + \sum_{j,k=1}^{2} \frac{\partial^2 \rho(w)}{\partial z_j \partial \overline{z_k}} (z_j - w_j)(\overline{z_k} - \overline{w_k}) + o(||z - w||^2),$$

The strict plurisubharmonicity of ρ implies the existence of a $c' > 0$ so that

$$\sum_{j,k=1}^{2} \frac{\partial^2 \rho(w)}{\partial z_j \partial \overline{z_k}} (z_j - w_j)(\overline{z_k} - \overline{w_k}) \geq c'||z - w||^2,\ \ (z, w) \in \overline{\Omega} \times \overline{\Omega}.$$

The result follows quite easily from this. □

5.1 Special Darboux Coordinates

As we are now going to construct a non-holomorphic transformation, we need to alternate between the real and complex notation. Here are some clarifications.

- We will use z (and similarly w) to denote both $(z_1, z_2) = (x_1 + iy_1, x_2 + iy_2) \in \mathbb{C}^2$ and $(x_1, y_1, x_2, y_2) \in \mathbb{R}^4$. The usage will be clear from the context. In the same vein, by z' we mean either $(z_1, x_2) = (x_1 + iy_1, x_2) \in \mathbb{C} \times \mathbb{R}$ or $(x_1, y_1, x_2) \in \mathbb{R}^3$.

- For any map $\Psi : \mathbb{C}^2 \to \mathbb{C}^2$ and $z, w \in \mathbb{C}^2$, $\mathrm{J}_{\mathbb{R}}\,\Psi(w)(z-w)$ will either denote a vector in $\mathbb{C}^2$ or a vector in $\mathbb{R}^4$ depending on the context. Recall that $\mathrm{J}_{\mathbb{R}}\,\Psi(w)(z-w) = \mathrm{J}_{\mathbb{C}}\,\Psi(w)(z-w) + \mathrm{Jac}_{\overline{\mathbb{C}}}\,\Psi(w)(\overline{z}-\bar{w})$, where $\mathrm{Jac}_{\overline{\mathbb{C}}}\,\Psi(w)$ is the matrix of complex conjugate derivatives of Ψ at w.
- Recall that $\left\langle \theta, z \right\rangle$ denotes the pairing between a complex covector and a complex vector. When θ is a real covector, we use $\left\langle\!\left\langle \theta, z \right\rangle\!\right\rangle$ to stress that z is being viewed as a tuple in $\mathbb{R}^4$.

Fix a $\lambda > 0$. For reasons that will become clear in the next part of this section, we consider a special $\mathcal{C}^4$-smooth strongly pseudoconvex domain Ω such that $0 \in \partial\Omega$ and for a neighborhood U of the origin, there is a convex function $\rho : U \to \mathbb{R}$ such that $\Omega \cap U = \{z \in U : \rho(z) < 0\}$ and

$$\rho(z) = -\operatorname{Im} z_2 + \lambda|z_1|^2 + 2\operatorname{Re}(\mu z_1 \overline{z_2}) + \nu|z_2|^2 + o(|z|^2). \tag{5.2}$$

We may shrink U to find a convex function $F := F_\rho : U' \to \mathbb{R}$ that satisfies $\rho(z_1, x_2, F(z_1, x_2)) = 0$. ρ and F_ρ are both $\mathcal{C}^4$-smooth and $-i(\partial\rho - \overline{\partial}\rho)$ is a $\mathcal{C}^3$-smooth contact form on $\partial\Omega \cap U$. The domain $\mathcal{S}_\lambda$ from Sect. 4 is such a domain with $\rho^\lambda(z) = -\operatorname{Im} z_2 + \lambda|z_1|^2$ and $F_{\rho^\lambda}(z_1, x_2) = \lambda|z_1|^2$.

Darboux's theorem in contact geometry (see [1, Appendix 4]) says that any two equi-dimensional contact structures are locally contactomorphic. We seek local diffeomorphisms between Ω and $\mathcal{S}_\lambda$ that extend to local contactomorphisms between $(\partial\Omega, -i(\partial\rho - \overline{\partial}\rho))$ and $(\partial\mathcal{S}_\lambda, -i(\partial\rho^\lambda - \overline{\partial}\rho^\lambda))$, and satisfy estimates essential to our goal. We carry out this construction over the next three lemmas, working initially on $\mathbb{R}^3$ instead of $\partial\Omega$. For this, if $\mathsf{gr}_\rho : U' \to U$ maps (x_1, y_1, x_2) to $(x_1, y_1, x_2, F_\rho(x_1, y_1, x_2))$, we set

$$\begin{aligned}\theta_\rho &:= (\mathsf{gr}_\rho)^*\left(\frac{\partial\rho - \overline{\partial}\rho}{i}\right)\\ &= \frac{-1}{\rho_{y_2}}\Big((\rho_{y_2}\rho_{y_1} + \rho_{x_1}\rho_{x_2})dx_1 - (\rho_{y_2}\rho_{x_1} - \rho_{y_1}\rho_{x_2})dy_1 + (\rho_{y_2}^2 + \rho_{x_2}^2)dx_2\Big),\end{aligned}$$

where, by the partial derivatives of ρ we mean their pull-backs to U' via gr_ρ.

Lemma 5.2 *Let Ω be defined by (5.2). There is an open subset $(0 \in) V \subset U' \subset \mathbb{R}^3$ and a $\mathcal{C}^2$-smooth diffeomorphism $\Pi = (\pi_1, \pi_2, \pi_3) : V \to \mathbb{R}^3$ with $\Pi(0) = 0$ satisfying*

- $\Pi^*\theta_{\rho^\lambda}(z') = \alpha(z')\theta_\rho(z')$ *for all $z' \in V$, and some $\alpha \in \mathcal{C}(V)$ with $\alpha(0) = 1$; and*
- $|\det \mathrm{J}_{\mathbb{R}}\,\Pi(0)| = 1$.

Proof We proceed with the understanding that when referring to functions defined a priori on U (such as ρ or its derivatives) we implicitly mean their pull-backs to U' via gr_ρ.

Now, consider the following $\mathcal{C}^3$-smooth vector field in $\ker \theta_\rho$ on U':

$$v = \frac{\partial \rho}{\partial x_2}\frac{\partial}{\partial x_1} - \frac{\partial \rho}{\partial y_2}\frac{\partial}{\partial y_1} - \frac{\partial \rho}{\partial x_1}\frac{\partial}{\partial x_2}.$$

We let $\gamma^t(z') := \gamma(z';t) = (\gamma_1(z';t), \gamma_2(z';t), \gamma_3(z';t))$ be the flow of v such that $\gamma(z';0) = z'$. Note that $\gamma(z';t)$ is $\mathcal{C}^3$-smooth in z' and $\mathcal{C}^4$-smooth in t. Differentiating the initial value problem for the flow, we have

$$\mathrm{J}_{\mathbb{R}}\,\gamma^0 \equiv \mathrm{Id.} \text{ and } \mathrm{Hess}_{\mathbb{R}}\,\gamma^0 \equiv 0. \tag{5.3}$$

Observe that the map

$$\Gamma = (\Gamma_1, \Gamma_2, \Gamma_3) : z' = (x_1, y_1, x_2) \mapsto \gamma(x_1, 0, x_2; y_1),$$

is defined on some neighborhood, $U'_1 \subset U'$, of the origin. Moreover, dropping the arguments, switching to our shorthand notation, and denoting $f \circ \Gamma$ by $\widetilde{f}$, we have

$$\mathrm{J}_{\mathbb{R}}\,\Gamma = \begin{pmatrix} \Gamma_{1x_1} & \widetilde{\rho_{x_2}} & \Gamma_{1x_2} \\ \Gamma_{2x_1} & -\widetilde{\rho_{y_2}} & \Gamma_{2x_2} \\ \Gamma_{3x_1} & -\widetilde{\rho_{x_1}} & \Gamma_{3x_2} \end{pmatrix},$$

and

$$(\mathrm{J}_{\mathbb{R}}\,\Gamma)^{-1} = \begin{pmatrix} \frac{\widetilde{\rho_{x_1}}\Gamma_{2x_2} - \widetilde{\rho_{y_2}}\Gamma_{3x_2}}{\det \mathrm{J}_{\mathbb{R}}\,\Gamma} & \frac{-\widetilde{\rho_{x_1}}\Gamma_{1x_2} - \widetilde{\rho_{x_2}}\Gamma_{3x_2}}{\det \mathrm{J}_{\mathbb{R}}\,\Gamma} & \frac{\widetilde{\rho_{y_2}}\Gamma_{1x_2} + \widetilde{\rho_{x_2}}\Gamma_{2x_2}}{\det \mathrm{J}_{\mathbb{R}}\,\Gamma} \\ \frac{\Gamma_{2x_2}\Gamma_{3x_1} - \Gamma_{2x_1}\Gamma_{3x_2}}{\det \mathrm{J}_{\mathbb{R}}\,\Gamma} & \frac{-\Gamma_{1x_2}\Gamma_{3x_1} + \Gamma_{1x_1}\Gamma_{3x_2}}{\det \mathrm{J}_{\mathbb{R}}\,\Gamma} & \frac{\Gamma_{1x_2}\Gamma_{2x_1} - \Gamma_{1x_1}\Gamma_{2x_2}}{\det \mathrm{J}_{\mathbb{R}}\,\Gamma} \\ \frac{\widetilde{\rho_{y_2}}\Gamma_{3x_1} - \widetilde{\rho_{x_1}}\Gamma_{2x_1}}{\det \mathrm{J}_{\mathbb{R}}\,\Gamma} & \frac{\widetilde{\rho_{x_2}}\Gamma_{3x_1} + \widetilde{\rho_{x_1}}\Gamma_{1x_1}}{\det \mathrm{J}_{\mathbb{R}}\,\Gamma} & \frac{-\widetilde{\rho_{y_2}}\Gamma_{1x_1} - \widetilde{\rho_{x_2}}\Gamma_{2x_1}}{\det \mathrm{J}_{\mathbb{R}}\,\Gamma} \end{pmatrix},$$

wherever $\mathrm{J}_{\mathbb{R}}\,\Gamma$ is invertible. In particular, $\mathrm{J}_{\mathbb{R}}\,\Gamma(0) = (\mathrm{J}_{\mathbb{R}}\,\Gamma)^{-1}(0) = \mathrm{Id}$. We may, therefore, locally invert Γ (as a $\mathcal{C}^3$-smooth function) in some neighborhood $W_1 \subset U'_1$ of 0. Let

$$(X_1, Y_1, X_2) = \Gamma^{-1}(x_1, y_1, x_2).$$

Γ is constructed to "straighten" v, i.e., $\mathrm{J}_{\mathbb{R}}\,\Gamma(\frac{\partial}{\partial Y_1}) = v$. So, if we view X_1 and X_2 as $\mathcal{C}^3$-smooth functions on $W := \Gamma(W_1) \cap U'$, they have linearly independent differentials and $v(X_1) \equiv v(X_2) \equiv 0$. Thus, $dX_1 \wedge dX_2 \neq 0$ everywhere on W and $dX_1(v) \equiv dX_2(v) \equiv \theta_\rho(v) \equiv 0$ on W. So, it must be the case that

$$\theta_\rho(\cdot) = \omega_1(\cdot)dX_1(\cdot) + \omega_2(\cdot)dX_2(\cdot),$$

for some $\omega_1, \omega_2 \in \mathcal{C}^2(W)$. Substituting the expressions for θ_ρ, dX_1 and dX_2 (the latter two can be read off the matrix $(\mathrm{J}_{\mathbb{R}}\,\Gamma)^{-1}$ above), we get $\omega_1 =$

$$\frac{-\Gamma_{1x_1}\widetilde{\rho_{y_2}}(\rho_{y_1}\rho_{y_2} + \rho_{x_1}\rho_{x_2}) + \Gamma_{2x_1}(\widetilde{\rho_{x_1}}(\rho_{y_2}^2 + \rho_{x_2}^2) - \widetilde{\rho_{x_2}}(\rho_{y_1}\rho_{y_2} + \rho_{x_1}\rho_{x_2})) - \Gamma_{3x_1}\widetilde{\rho_{y_2}}(\rho_{x_2}^2 + \rho_{y_2}^2)}{\rho_{y_2}\widetilde{\rho_{y_2}}}$$

and $\omega_2 =$

$$\frac{-\Gamma_{1x_2}\widetilde{\rho_{y_2}}(\rho_{y_1}\rho_{y_2}+\rho_{x_1}\rho_{x_2})+\Gamma_{2x_2}(\widetilde{\rho_{x_1}}(\rho_{y_2}^2+\rho_{x_2}^2)-\widetilde{\rho_{x_2}}(\rho_{y_1}\rho_{y_2}+\rho_{x_1}\rho_{x_2}))-\Gamma_{3x_2}\widetilde{\rho_{y_2}}(\rho_{x_2}^2+\rho_{y_2}^2)}{\rho_{y_2}\widetilde{\rho_{y_2}}},$$

where, once again, $\widetilde{f} := f \circ \Gamma$. Observe that $\omega_1(0) = 0$ and $\omega_2(0) = 1$. Thus, for some neighborhood, $V \subset W$, of the origin, $\omega_2 \neq 0$ and

$$\theta_\rho = \omega_2(Y_1 dX_1 + dX_2),$$

where $Y_1 := \omega_1/\omega_2$. Finally, set

$$\alpha := \frac{1}{\omega_2},\ \pi_1 := X_1,\ \pi_2 := -\frac{Y_1}{4\lambda} \text{ and } \pi_3 := X_2 + \frac{X_1 Y_1}{2}.$$

Then, on V,

$$\alpha\theta_\rho = -2\lambda\pi_2 d\pi_1 + 2\lambda\pi_1 d\pi_2 + d\pi_3 = \Pi^*(\theta_{\rho^\lambda}) \tag{5.4}$$

and $\alpha(0) = 1$.

Referring to (5.3) and the formulae for ω_1, ω_2 and $(\mathrm{J}_\mathbb{R}\,\Gamma)^{-1}$, we get

$$\mathrm{J}_\mathbb{R}\,\Pi(0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{\operatorname{Im}\mu}{2\lambda} \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.5}$$

We have, thus, constructed the required map. □

We now show that the contact transformation constructed above satisfies an estimate crucial to our analysis.

Lemma 5.3 *Let Π and V be as in the proof of Lemma 5.2 and $\mathcal{V} \Subset V$ be a neighborhood of the origin. Then, there is an $e_1 \in \mathcal{C}(\mathcal{V})$ with $e_1(0) = 0$ and a $c_1 > 0$ such that, for all $w' \in \mathcal{V}$ and $z' \in \mathbb{R}^3$,*

$$\begin{aligned} &|(z'-w')^{\mathrm{tr}} \cdot \mathrm{Hess}_\mathbb{R}\,\pi_3(w') \cdot (z'-w')| \\ &\quad \le e_1(w')|z'-w'|^2 + c_1(|z_1-w_1||x_2-u_2| + |x_2-u_2|^2). \end{aligned} \tag{5.6}$$

Proof Recall that $\pi_3 = X_2 + \frac{X_1 Y_1}{2}$. We refer to the construction in the proof of Lemma 5.2 and collect the following data:

$$\begin{aligned} &(X_1)_{x_1}(0) = 1,\ (X_1)_{y_1}(0) = 0; \\ &(Y_1)_{x_1}(0) = 0,\ (Y_1)_{y_1}(0) = -4\lambda; \\ &(X_2)_{x_1x_1}(0) = 0,\ (X_2)_{x_1y_1}(0) = 2\lambda = (X_2)_{y_1x_1}(0),\ (X_2)_{y_2y_2}(0) = 0. \end{aligned}$$

Next, we write out the relevant terms.

$(z'-w')^{\mathrm{tr}} \cdot \mathrm{Hess}_\mathbb{R}\,\pi_3(w') \cdot (z'-w')$

$$= \Big(X_{2x_1x_1}(w') + X_{1x_1}(w')Y_{1x_1}(w') + \frac{1}{2}Y_1(w')X_{1x_1x_1}(w') + \frac{1}{2}X_1(w')Y_{1x_1x_1}(w')\Big)(x_1-u_1)^2$$
$$+ \Big(2X_{2x_1y_1}(w') + X_{1x_1}(w')Y_{1y_1}(w') + X_{1y_1}(w')Y_{1x_1}(w')\Big)(x_1-u_1)(y_1-v_1)$$
$$+ \Big(Y_1(w')X_{1x_1y_1}(w') + X_1(w')Y_{1x_1y_1}(w')\Big)(x_1-u_1)(y_1-v_1)$$
$$+ \Big(X_{2y_1y_1}(w') + X_{1y_1}(w')Y_{1y_1}(w') + \frac{1}{2}Y_1(w')X_{1y_1y_1}(w') + \frac{1}{2}X_1(w')Y_{1y_1y_1}(w')\Big)(y_1-v_1)^2$$
$$+ 2\pi_{3x_1x_2}(w')(x_1-u_1)(x_2-u_2) + 2\pi_{3y_1x_2}(w')(y_1-v_1)(x_2-u_2) + \pi_{3x_2x_2}(w')(x_2-u_2)^2.$$

Now, the coefficients of $(x_1-u_1)^2$, $(x_1-u_1)(y_1-v_1)$ and $(y_1-u_1)^2$ in the above expansion all vanish at the origin (see data listed above). Thus, we obtain (5.6). □

All that remains is to extend the above transformation to Ω. For this, let V be as in Lemma 5.2 and $G_\rho : V \times \mathbb{R} \to \mathbb{C}^2$ be the map

$$(x_1, y_1, x_2, y_2) \mapsto (x_1, y_1, x_2, F_\rho(x_1, y_1, x_2) + y_2).$$

G_ρ is evidently a $\mathcal{C}^4$-smooth diffeomorphism with $G(V \times (0, t]) \subset \Omega$ for some $t > 0$. We note the following facts about G_ρ:

- $\mathrm{J}_{\mathbb{R}}\, G_\rho(0) = \mathrm{Id}$.
- $(G_\rho)^*(d\rho) = \left(\frac{\partial\rho}{\partial y_2} \circ G_\rho\right) dy_2$ and $(G_\rho)^*\left(\frac{\partial\rho - \overline{\partial}\rho}{i}\right) = \theta_\rho$ on $V \times \{0\}$.

Lemma 5.4 *There is a neighborhood $U \subset \mathbb{C}^2$ of the origin and a $\mathcal{C}^2$-diffeomorphism $\Psi : U \to \Psi(U) \subset \mathbb{C}^2$ such that*

- *$\Psi(0) = 0$, $\Psi(\Omega \cap U) = \mathcal{S}_\lambda \cap \Psi(U)$ and $\Psi(\partial\Omega \cap U) = \partial\mathcal{S}_\lambda \cap \Psi(U)$;*
- $\det \mathrm{J}_{\mathbb{R}}\, \Psi(0) = 1$ *and* $\det \mathrm{J}_{\mathbb{R}}\, \Psi^{\mathrm{res}}(0) = 1$*; and*
- *if $\mathfrak{l}_\rho$ and $\mathfrak{l}_\lambda$ denote the Cauchy–Leray map of ρ and ρ^λ, respectively, then*

$$\begin{aligned}&\left|\mathfrak{l}_\rho(z, w) - \mathfrak{l}_\lambda(\Psi(z), \Psi(w))\right| \\ &\quad\le (e(w) + D(z-w))\left(|\mathfrak{l}_\lambda(\Psi(z), \Psi(w)) + ||z-w||^2\right) + c|\mathfrak{l}_\lambda(\Psi(z), \Psi(w))|^2,\end{aligned} \tag{5.7}$$

on $\{(z, w) \in \overline{\Omega} \times U : ||z-w|| \le \tau\}$, for some choice of $e \in \mathcal{C}(U)$ with $e(0) = 0$, $D(\zeta) = o(1)$ as $|\zeta| \to 0$, and constants $c, \tau > 0$.

Proof Let $\Psi = (\Psi_1, \Psi_2) := G_{\rho^\lambda} \circ (\Pi, \mathrm{Id.}) \circ G_\rho^{-1}$, where Id. is the identity map on $\mathbb{R}$, and $U \Subset G_\rho(V \times [-t, t])$ is a neighborhood of the origin. We use the notation $(\Psi_1, \Psi_2) = (\psi_1 + i\psi_2, \psi_3 + i\psi_4)$. The regularity and mapping properties of Ψ follow from its definition We also clarify that $\partial\rho^\lambda(\Psi(w))$ denotes $\partial\rho^\lambda$ evaluated at $\Psi(w)$. Since $\mathrm{Id.}^*(-dy_2) = -dy_2$ and $\Pi^*(\theta_{\rho^\lambda}) = \alpha\theta_\rho$ on $\{y_2 = 0\}$,

$$\Psi^*(d\rho^\lambda) = \alpha_1(d\rho)$$

and

$$\Psi^*\left(\frac{\partial\rho^\lambda - \overline{\partial}\rho^\lambda}{i}\right) = \alpha_2\left(\frac{\partial\rho - \overline{\partial}\rho}{i}\right),$$

on $\partial\Omega \cap U$, where $\alpha_1(x_1, y_1, x_2, y_2) = -1/\left(\dfrac{\partial\rho}{\partial y_2}(G_\rho(x_1, y_1, x_2, y_2))\right)$ and $\alpha_2(x_1, y_1, x_2, y_2) = \alpha(x_1, y_1, x_2)$. Therefore, for all $w \in \partial\Omega \cap U$ and $z \in \mathbb{C}^2$,

$$
\begin{aligned}
&2\left\langle \partial\rho^\lambda(\Psi(w)), \mathrm{J}_{\mathbb{R}}\,\Psi(w)(z-w)\right\rangle \qquad (5.8)\\
&= 2\,\mathrm{Re}\left\langle \partial\rho^\lambda(\Psi(w)), \mathrm{J}_{\mathbb{R}}\,\Psi(w)(z-w)\right\rangle + 2i\,\mathrm{Im}\left\langle \partial\rho^\lambda(\Psi(w)), \mathrm{J}_{\mathbb{R}}\,\Psi(w)(z-w)\right\rangle\\
&= \left\langle\!\left\langle (\partial\rho^\lambda + \overline{\partial}\rho^\lambda)(\Psi(w)), \mathrm{J}_{\mathbb{R}}\,\Psi(w)(z-w)\right\rangle\!\right\rangle\\
&\quad + i\left\langle\!\left\langle \frac{\partial\rho^\lambda - \overline{\partial}\rho^\lambda}{i}(\Psi(w)), \mathrm{J}_{\mathbb{R}}\,\Psi(w)(z-w)\right\rangle\!\right\rangle\\
&= \left\langle\!\left\langle \Psi^*(\partial\rho^\lambda + \overline{\partial}\rho^\lambda)(w), z-w\right\rangle\!\right\rangle + i\left\langle\!\left\langle \Psi^*\left(\frac{\partial\rho^\lambda - \overline{\partial}\rho^\lambda}{i}\right)(w), z-w\right\rangle\!\right\rangle\\
&= \alpha_1(w)\left\langle\!\left\langle (\partial\rho + \overline{\partial}\rho)(w), z-w\right\rangle\!\right\rangle + i\alpha_2(w)\left\langle\!\left\langle \left(\frac{\partial\rho - \overline{\partial}\rho}{i}\right)(w), z-w\right\rangle\!\right\rangle\\
&= 2\alpha_1(w)\,\mathrm{Re}\left\langle \partial\rho(w), z-w\right\rangle + 2i\alpha_2(w)\,\mathrm{Im}\left\langle \partial\rho(w), z-w\right\rangle.
\end{aligned}
$$

Now, since $\rho^\lambda := \lambda|z_1|^2 - y_2$, $\dfrac{\partial\rho^\lambda}{\partial z_1}(\Psi(z)) = \lambda\overline{\Psi_1(z)}$ and $\dfrac{\partial\rho^\lambda}{\partial z_2}(\Psi(z)) = \dfrac{i}{2}$. Therefore, there is a $\tau_1 > 0$ such that on $\{(z, w) \in \mathbb{R}^4 \times U : ||z-w|| \le \tau_1\}$,

$$
\begin{aligned}
&\left|\left\langle \partial\rho^\lambda(\Psi(w)), \Psi(z) - \Psi(w) - \mathrm{J}_{\mathbb{R}}\,\Psi(w)(z-w)\right\rangle\right|\\
&\quad \le c|\Psi_1(w)| \cdot ||z-w||^2 + \frac{1}{2}R_1(z-w) + R_2(z-w), \qquad (5.9)
\end{aligned}
$$

where, $c' > 0$, $R_1(z-w) = |(z-w)^{\mathrm{tr}} \cdot (\mathrm{Hess}_{\mathbb{R}}\,\psi_3(w) + \mathrm{Hess}_{\mathbb{R}}\,\psi_4(w)) \cdot (z-w)|$, and $R_2(\zeta) = o(|\zeta|^2)$ as $|\zeta| \to 0$. Observe that $\psi_3(z', y_2) = \pi_3(z')$ and $\psi_4(z', y_2) = \pi_1(z')^2 + \pi_2(z')^2 + y_2 - F(z')$. As

$$
\begin{aligned}
\psi_{4_{x_1x_1}}(w) &= 2\sum_{j=1}^{2}(\pi_{j_{x_1}}(w')^2 + \pi_j(w')\pi_{j_{x_1x_1}}(w')) - F_{x_1x_1}(w'),\\
\psi_{4_{y_1y_1}}(w) &= 2\sum_{j=1}^{2}(\pi_{j_{y_1}}(w')^2 + \pi_j(w')\pi_{j_{y_1y_1}}(w')) - F_{y_1y_1}(w') \text{ and}\\
\psi_{4_{x_1y_1}}(w) &= 2\sum_{j=1}^{2}(\pi_{j_{x_1}}(w')\pi_{j_{y_1}}(w') + \pi_j(w')\pi_{j_{x_1y_1}}(w')) - F_{x_1y_1}(w')
\end{aligned}
$$

all vanish at $w = 0$, we have, for all $(z, w) \in \mathbb{R}^4 \times U$,

$$|(z-w)^{\mathrm{tr}} \cdot \mathrm{Hess}_{\mathbb{R}}\,\psi_4(w) \cdot (z-w)|$$

$$\leq e_2(w)||z-w||^2 + c_2(|z_1-w_1||z_2-w_2| + |z_2-w_2|^2), \quad (5.10)$$

where $e_1 \in \mathcal{C}(U)$ with $e_1(0) = 0$, and $c_1 > 0$ is a constant. Combining (5.9), (5.6) and (5.10) (and adding $c'|\Psi_1|$, e_1 and e_2), we have that

$$\mathcal{A} := \left|\left\langle \partial\rho^\lambda(\Psi(w)), \Psi(z) - \Psi(w) - \mathrm{J}_{\mathbb{R}}\,\Psi(w)(z-w)\right\rangle\right| \quad (5.11)$$
$$\leq (e_3(w) + D_3(z-w))||z-w||^2 + c_3(|z_1-w_1||z_2-w_2| + |z_2-w_2|^2),$$

on $\{(z,w) \in \mathbb{R}^4 \times U : ||z-w|| \leq \tau_3\}$, for some $e_3 \in \mathcal{C}(U)$ with $e_3(0) = 0$, $D_3(\zeta) = o(1)$ as $|\zeta| \to 0$, and constants $c_3, \tau_3 > 0$.

Next, we have that

$$|\Psi_2(z) - \Psi_2(w)| = 2\left|\left\langle \partial\rho^\lambda(\Psi(w)), \Psi(z) - \Psi(w)\right\rangle - \overline{\Psi_1(z)}(\Psi_1(z) - \Psi_1(w))\right|$$
$$\leq c_4\left|\left\langle \partial\rho^\lambda(\Psi(w)), \Psi(z) - \Psi(w)\right\rangle\right| + e_4(w)||z-w||, \quad (5.12)$$

on $\{(z,w) \in \mathbb{R}^4 \times U : ||z-w|| \leq \tau_4\}$, for some choice of e_4, c_4 and τ_4 as before. Also, if $\Psi^{-1} = (\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, \hat{\psi}_4)$, then $\mathrm{J}_{\mathbb{R}}\,\hat{\psi}_3(0) = (0,0,1,0)$ and $\mathrm{J}_{\mathbb{R}}\,\hat{\psi}_4(0) = (0,0,0,1)$. So, we are permitted to conclude that

$$|z_2 - w_2| \leq c_4|\Psi_2(z) - \Psi_2(w)| + (e_5(w) + D_5(z-w))||z-w||, \quad (5.13)$$

on $\{(z,w) \in \mathbb{R}^4 \times U : ||z-w|| \leq \tau_5\}$, for some e_5, c_5, D_5 and τ_5 as before.

Finally, as $\alpha_1(0) = \alpha_2(0) = 1$, (5.8), (5.11), (5.12) and (5.13) combine to give e, c, D and τ with the required properties, such that

$$|\mathfrak{l}_\rho(z,w) - \mathfrak{l}_\lambda(\Psi(z), \Psi(w))|$$
$$\leq \left|\left\langle \partial\rho(w), z-w\right\rangle - \left\langle \partial\rho^\lambda(\Psi(w)), \mathrm{J}_{\mathbb{R}}\,\Psi(w)(z-w)\right\rangle\right| + \mathcal{A},$$
$$\leq (e(w) + D(z-w))\left(|\mathfrak{l}_\lambda(\Psi(z), \Psi(w))| + ||z-w||^2\right) + c|\mathfrak{l}_\lambda(\Psi(z), \Psi(w))|^2$$

on $\{(z,w) \in \mathbb{R}^4 \times U : ||z-w|| \leq \tau\}$. □

Convexification Now, we return to general strongly pseudoconvex domains. Assume $0 \in \partial\Omega$ and the outward unit normal vector to $\partial\Omega$ at 0 is $(0, -i)$. Let ρ be a $\mathcal{C}^2$-smooth strictly plurisubharmonic defining function of Ω such that $||\nabla\rho(0)|| = 1$. Now, ρ has the following second-order Taylor expansion about the origin:

$$\rho(w) = \mathrm{Im}\left(-w_2 + i\sum_{j,k=1}^{2} \frac{\partial^2\rho(0)}{\partial z_j \partial z_k} w_j w_k\right) + \sum_{j,k=1}^{2} \frac{\partial^2\rho(0)}{\partial z_j \partial \overline{z_k}} w_j \overline{w_k} + o(|w|^2).$$

Using a classical trick, attributed to Narasimhan, we convexify Ω near the origin via the map Φ given by:

$$
\begin{aligned}
w_1 &\mapsto \Phi_1(w) = w_1 \\
w_2 &\mapsto \Phi_2(w) = w_2 - i \sum_{j,k=1}^{2} \frac{\partial^2 \rho(0)}{\partial z_j \partial z_k} w_j w_k.
\end{aligned}
$$

Owing to the inverse function theorem, Φ is a local biholomorphism on some neighborhood U of 0. We may further shrink U so that the strong convexity of $\Phi(\partial\Omega \cap U)$ at 0 propagates to all of $\Phi(\partial\Omega \cap U)$. We collect the following key observations:

- $\mathrm{J}_{\mathbb{R}}\, \Phi(0) = \mathrm{Id.}$;
- If $\hat{\rho} := \rho \circ \Phi^{-1}$, then $\hat{\rho}(w) = -\operatorname{Im} w_2 + \sum_{j,k=1}^{2} \frac{\partial^2 \rho(0)}{\partial z_j \partial \overline{z_k}} w_j \overline{w_k} + o(|w|^2)$.
- Let $\mathfrak{p}$ denote the Levi-polynomial of ρ, $\mathfrak{l}_{\hat{\rho}}(z, w)$ be the Cauchy–Leray map of $\hat{\rho}$, and $\partial\hat{\rho}(\Phi(w))$ denote $\partial\hat{\rho}$ evaluated at $\Phi(w)$. Then, for any neighborhood $U_1 \Subset U$ of the origin, there is a $\tau > 0$ such that, on $\{(z, w) \in \mathbb{C}^2 \times U_1 : ||z - w|| \leq \tau\}$,

$$
\begin{aligned}
|\mathfrak{p}(z, w) - \mathfrak{l}_{\hat{\rho}}(\Phi(z), \Phi(w))| &= |\mathfrak{p}(z, w) - \mathfrak{l}_{\hat{\rho}}(\Phi(z), \Phi(w))| \qquad (5.14)\\
&\leq \left| \left\langle \partial\rho(w), z - w \right\rangle - \left\langle \partial\hat{\rho}(\Phi(w)), \mathrm{J}_{\mathbb{R}}\, \Phi(w)(z - w) \right\rangle \right| \\
&\quad + \frac{1}{2} \left| \sum_{j,k=1}^{2} \left(\frac{\partial^2 \rho(w)}{\partial z_j \partial z_k} + 2i \frac{\partial\hat{\rho}(\Phi(w))}{\partial w_2} \frac{\partial^2 \rho(0)}{\partial z_j \partial z_k} \right) (z_j - w_j)(z_k - w_k) \right| \\
&\leq \left| \left\langle \partial\rho(w), z - w \right\rangle - \left\langle \Phi^*(\partial\hat{\rho})(w), z - w \right\rangle \right| \\
&\quad + \frac{1}{2} \left| \sum_{j,k=1}^{2} \left(\frac{\partial^2 \rho(0)}{\partial z_j \partial z_k} + o(1) + (-1 + o(|w|)) \frac{\partial^2 \rho(0)}{\partial z_j \partial z_k} \right) (z_j - w_j)(z_k - w_k) \right| \\
&\leq e(w) ||z - w||^2,
\end{aligned}
$$

  for some $e \in \mathcal{C}(U)$ with $e(0) = 0$.

Main Local Estimate We combine the maps constructed above:

Lemma 5.5 *Fix an $\varepsilon > 0$. Let $\Omega \subset \mathbb{C}^2$ be a $\mathcal{C}^4$-smooth strongly pseudoconvex domain and ρ a strictly plurisubharmonic defining function of Ω. Assume that $0 \in \partial\Omega$, $\nabla\rho(0) = (0, 0, 0, -1)$ and $M(\rho)(0) = \lambda$. Then, there exists a neighborhood U of the origin, a $\mathcal{C}^2$-smooth origin-preserving diffeomorphism Θ on U that carries $\overline{\Omega} \cap U$ onto $\overline{\mathcal{S}_\lambda} \cap \Theta(U)$, and a constant $\tau > 0$ such that*

- $1 - \varepsilon \leq \frac{\mathrm{vol}(\Theta(V))}{\mathrm{vol}(V)} \leq \frac{1}{1 - \varepsilon}$, *for every Jordan measurable $V \subset U$;*
- $1 - \varepsilon \leq \frac{\mathrm{vol}_3(\Theta(J)')}{\mathrm{vol}_3(J')} \leq \frac{1}{1 - \varepsilon}$, *for every Jordan measurable $J \subset \partial\Omega \cap U$; and*

- *if* $\mathfrak{p}$ *is the Levi polynomial of* ρ *and* $\mathfrak{l}_\lambda$ *is the Cauchy–Leray map of* ρ^λ*, then*

$$|\mathfrak{p}(z,w) - \mathfrak{l}_\lambda(\Theta(z),\Theta(w))| \le \varepsilon(|\mathfrak{p}(z,w)| + |\mathfrak{l}_\lambda(\Theta(z),\Theta(w))|)$$

  on $(U \times U) \cap \Omega_\tau$.

Proof The needed map is $\Psi \circ \Phi$ (from Lemma 5.4 and the convexification procedure above). The mapping and volume distortion properties follow from those of Ψ and Φ. The estimate is a combination of (5.14), (5.7) and (5.1). □

The following lemma is an application of Lemma 3.2 and gives us a local version of our main theorem.

Lemma 5.6 *Let* Ω*,* f *and* ρ *be as in Theorem 1.1. Fix an* $\varepsilon \in (0, 1/3)$ *and a point* $q \in \partial\Omega$*. Then, there exists a neighborhood* $U_{q,\varepsilon}$ *of* q *such that for every Jordan measurable pair* $J, H \subset \partial\Omega \cap U_{q,\varepsilon}$ *such that* $J \subset \mathrm{int}_{\partial\Omega} H$*,*

$$(1-\varepsilon)^{31}\, l_{\mathrm{kor}} \frac{\lambda(q)^{\frac{1}{2}} s(J)^{\frac{3}{2}}}{\sqrt{n}} \le v(\Omega; \mathcal{P}_n(J \subset H; f)) \le (1-\varepsilon)^{-19}\, l_{\mathrm{kor}} \frac{\lambda(q)^{\frac{1}{2}} s(J)^{\frac{3}{2}}}{\sqrt{n}}$$

for sufficiently large n*, where* $\lambda(q) := \dfrac{4M(\rho)(q)}{||\nabla\rho(q)||^3}$ *and* s *is the Euclidean surface area measure on* $\partial\Omega$*.*

Proof First, we set $\hat{\varepsilon} = c\varepsilon$, where $c < 1$ will be revealed later. Let ρ be the strictly plurisubharmonic defining function of Ω for which (1.3) in Theorem 1.1 holds. Let $A : \mathbb{C}^2 \to \mathbb{C}^2$ be a holomorphic isometry that takes q to the origin and the outer unit normal at q to $(0, -i||\nabla\rho(q)||)$. Set $\hat{\rho}(z) := ||\nabla\rho(q)||^{-1}\rho(A^{-1}z)$. Then, $A(\Omega)$ and $\hat{\rho}$ satisfies the hypotheses of Lemma 5.5, with $M(\hat{\rho})(0) = \lambda(q)$. Suppose Θ, U and τ are the map, neighborhood and constant, respectively, granted by Lemma 5.5, and $\hat{\mathfrak{p}}$ is the Levi polynomial of $\hat{\rho}$. Then,

$$|\hat{\mathfrak{p}}(z,w) - \mathfrak{l}_{\lambda(q)}(\Theta(z),\Theta(w))| \le \hat{\varepsilon}(|\hat{\mathfrak{p}}(z,w)| + |\mathfrak{l}_{\lambda(q)}(\Theta(z),\Theta(w))|) \tag{5.15}$$

on $(U \times U) \cap A(\Omega)_\tau$. Also note that

$$||\nabla\rho(q)||\hat{\mathfrak{p}}(Az, Aw) = \mathfrak{p}(z,w). \tag{5.16}$$

Next, set $U_q := A^{-1}(U)$ and $\Theta_q := \Theta \circ A$. Note that Θ_q maps $\overline{\Omega}$ to $\overline{\mathcal{S}}_{\lambda(q)}$ locally near q. We define

$$\tilde{f}(z,w) := \frac{f(z,w)}{||\nabla\rho(q)||}; \tag{5.17}$$

$$g(z,w) := f_{\mathcal{S}_{\lambda(q)}}\big(\Theta_q z, \Theta_q w\big); \text{ and} \tag{5.18}$$

$$\tilde{g}(z,w) := a(w,w)\mathfrak{l}_{\lambda(q)}\big(\Theta_q z, \Theta_q w\big) = a(w,w)\frac{i}{2\lambda(q)} f_{\mathcal{S}_{\lambda(q)}}\big(\Theta_q z, \Theta_q w\big). \tag{5.19}$$

So, when defined,

$$C(w,\delta;\tilde{f}) = C\left(w, ||\nabla\rho(q)||\delta; f\right); \text{ and} \tag{5.20}$$

$$C(w,\delta;\tilde{g}) = C\left(w, \frac{2\lambda(q)}{|a(w,w)|}\delta; g\right). \tag{5.21}$$

Thus, for our point of interest, there is little difference between f and $\tilde{f}$ (and, between g and $\tilde{g}$). Keeping this observation in mind, we will apply Lemma 3.2 to $\tilde{f}, \tilde{g} \in \mathcal{C}(\overline{\Omega} \times (\partial\Omega \cap U_q))$ (see Remark 3.3). To bound $|\tilde{f}(z,w) - \tilde{g}(z,w)|$ from above, we estimate $|\tilde{f}(z,w) - a(z,w)\hat{\mathfrak{p}}(Az,Aw)|$, $|a(z,w)\hat{\mathfrak{p}}(Az,Aw) - a(z,w)\mathfrak{l}_{\lambda(q)}\big(\Theta_q z, \Theta_q w\big)|$ and $|a(z,w)\mathfrak{l}_{\lambda(q)}\big(\Theta_q z, \Theta_q w\big) - \tilde{g}(z,w)|$.

By (1.3), we can find a $\tau_1 \in (0,\tau]$ such that

$$\begin{aligned} |\tilde{f}(z,w) - a(z,w)\hat{\mathfrak{p}}(Az,Aw)| &= \frac{|f(z,w) - a(z,w)\mathfrak{p}(z,w)|}{||\nabla\rho(q)||} \\ &\le \frac{\hat{\varepsilon}}{||\nabla\rho(q)||}|\mathfrak{p}(z,w)| \quad \text{on } \Omega_{\tau_1}. \end{aligned} \tag{5.22}$$

By Lemma 5.5, (5.16), (5.19) and the continuity of a on $\overline{\Omega}_\tau$, we shrink τ_1 so that on $(U_q \times U_q) \cap \Omega_{\tau_1}$,

$$\begin{aligned} &|a(z,w)\hat{\mathfrak{p}}(Az,Aw) - a(z,w)\mathfrak{l}_{\lambda(q)}\big(\Theta_q z, \Theta_q w\big)| \\ &\le |a(z,w)||\hat{\mathfrak{p}}(Az,Aw) - \mathfrak{l}_{\lambda(q)}\big(\Theta_q z, \Theta_q w\big)| \\ &\le \hat{\varepsilon}|a(z,w)|\left(|\hat{\mathfrak{p}}(Az,Aw)| + |\mathfrak{l}_{\lambda(q)}\big(\Theta_q z, \Theta_q w\big)|\right) \\ &= \hat{\varepsilon}|a(z,w)|\left(\frac{|\mathfrak{p}(z,w)|}{||\nabla\rho(q)||} + \frac{|\tilde{g}(z,w)|}{|a(w,w)|}\right) \\ &\le \hat{\varepsilon}\left(\frac{\max_{\Omega_\tau}|a(z,w)|}{||\nabla\rho(q)||}\right)|\mathfrak{p}(z,w)| + \hat{\varepsilon}\left(\frac{\max_{\Omega_\tau}|a(z,w)|}{\min_{\partial\Omega}|a(w,w)|}\right)|\tilde{g}(z,w)|, \end{aligned} \tag{5.23}$$

and

$$\begin{aligned} |a(z,w)\mathfrak{l}_{\lambda(q)}\big(\Theta_q z, \Theta_q w\big) - \tilde{g}(z,w)| &= |a(z,w) - a(w,w)| \cdot |\mathfrak{l}_{\lambda(q)}\big(\Theta_q z, \Theta_q w\big)| \\ &\le \frac{\hat{\varepsilon}}{\min_{\partial\Omega}|a(w,w)|}|\tilde{g}(z,w)|. \end{aligned} \tag{5.24}$$

Lastly, by (1.3), there exist $\tau_2 \in (0,\tau_1]$ and $l > 0$ such that

$$|\mathfrak{p}(z,w)| \le l|\tilde{f}(z,w)| \quad \text{on } \Omega_{\tau_2}. \tag{5.25}$$

Now, set

$$c = \frac{1}{2}\min\left\{1, \frac{||\nabla\rho(q)||}{l}, \frac{||\nabla\rho(q)||}{l\max_{\Omega_\tau}|a(z,w)|}, \frac{\min_{\partial\Omega}|a(w,w)|}{\max_{\Omega_\tau}|a(z,w)|}, \min_{\partial\Omega}|a(w,w)|\right\}.$$

Then, adding (5.22), (5.23) and (5.24), and using (5.25), we get

$$|\tilde{f}(z,w)-\tilde{g}(z,w)| \le \varepsilon\left(|\tilde{f}(z,w)|+|\tilde{g}(z,w)|\right) \qquad \text{on } (U_q\times U_q)\cap\Omega_{\tau_2}.$$

We now need to show that $\tilde{g}$ satisfies the remaining hypotheses of Lemma 3.2. But these are conditions on the cuts of $\tilde{g}$, which are identical to the cuts of g (by (5.21)). So, we work with g instead. Let $U_{q,\varepsilon} \Subset U_q$ be an open neighborhood of q, and $\delta_0 > 0$ be such that $C(w,\delta;g) \subset V_q$ for all $w \in U_{q,\varepsilon}\cap\partial\Omega$ and $\delta < \delta_0$. Then, there is a diffeomorphism

$$\Theta_q = \Theta\circ A : C(w,\delta;g) \to C\big(\Theta_q w,\delta; f_{\mathcal{S}_{\lambda(q)}}\big), \tag{5.26}$$

for $w \in U_{q,\varepsilon}\cap\partial\Omega$ and $\delta < \delta_0$. Therefore, exploiting Lemma 6.1, we get

(1) $C(w,\delta;g)$ is Jordan measurable for all $w \in U_{q,\varepsilon}\cap\partial\Omega$ and $\delta<\delta_0$;
(2) If $w^1,\dots,w^m \in U_{q,\varepsilon}\cap\partial\Omega$, $m\in\mathbb{N}_+$, then

$$\begin{aligned}
\operatorname{vol}\left(\bigcup_{j=1}^m C(w^j,(1+t)\delta;g)\right) &\le \frac{1}{1-\varepsilon}\operatorname{vol}\left(\bigcup_{j=1}^m C(\Theta_q w^j,(1+t)\delta; f_{\mathcal{S}_{\lambda(q)}})\right)\\
&\le \frac{(1+t)^3}{1-\varepsilon}\operatorname{vol}\left(\bigcup_{j=1}^m C(\Theta_q w^j,\delta; f_{\mathcal{S}_{\lambda(q)}})\right)\\
&\le \frac{(1+t)^3}{(1-\varepsilon)^2}\operatorname{vol}\left(\bigcup_{j=1}^m C(w^j,\delta;g)\right),
\end{aligned}$$

for all $t\in(0,16)$ and $\delta_j \le \delta_0/16$, $j=1,\dots,m$. Thus, g satisfies the doubling property (3.3) with quantifiers $\delta_g=\delta_0/16$ and $D(t)=(1-\varepsilon)^{-2}(1+t)^3$.

Lastly, we further shrink $U_{q,\varepsilon}$—if necessary—to ensure that for any s-measurable set $J\subset(U_{q,\varepsilon}\cap\partial\Omega)$,

$$1-\varepsilon \le \frac{s(J)}{\operatorname{vol}_3(J'')} \le \frac{1}{1-\varepsilon}, \tag{5.27}$$

where J'' is the orthogonal projection of J onto $T_q\partial\Omega$, and by $\operatorname{vol}_3(J'')$, we really mean $\operatorname{vol}_3(A(J''))$.

We are now ready to estimate. Consider Jordan measurable compact sets $J\subset H\subset (U_{q,\varepsilon}\cap\partial\Omega)$ such that $J\subset \operatorname{int}_{\partial\Omega}H$. By (5.20), (3.4) from Lemma 3.2, (5.21), the volume-distortion properties of Θ_q—see Lemma 5.5 and recall that A is an isometry—Corollary 4.3 and (5.27), we have that

$$
\begin{aligned}
&\limsup_{n\to\infty}\sqrt{n}\, v(\Omega;\mathcal{P}_n(J\subset H;\, f)) = \limsup_{n\to\infty}\sqrt{n}\, v(\Omega;\mathcal{P}_n(J\subset H;\,\tilde{f}))\\
&\leq \frac{1}{(1-\varepsilon)^2}\left(1+\frac{(1+\varepsilon)^2}{(1-\varepsilon)^2}-1\right)^3 \limsup_{n\to\infty}\sqrt{n}\, v(\Omega;\mathcal{P}_n(J\subset H;\,\tilde{g}))\\
&= \frac{1}{(1-\varepsilon)^2}\left(1+\frac{(1+\varepsilon)^2}{(1-\varepsilon)^2}-1\right)^3 \limsup_{n\to\infty}\sqrt{n}\, v(\Omega;\mathcal{P}_n(J\subset H;\, g))\\
&\leq (1-\varepsilon)^{-14}\limsup_{n\to\infty}\sqrt{n}\,(1-\varepsilon)^{-1} v(\mathcal{S}_{\lambda(q)};\mathcal{P}_n(\Theta_q J\subset \Theta_q H;\, f_{\mathcal{S}_{\lambda(q)}}))\\
&\leq (1-\varepsilon)^{-15}\, l_{\mathrm{kor}}\lambda(q)^{\frac{1}{2}}\,\mathrm{vol}_3((\Theta_q J)')^{\frac{3}{2}}\\
&\leq (1-\varepsilon)^{-\frac{33}{2}}\, l_{\mathrm{kor}}\lambda(q)^{\frac{1}{2}}\,\mathrm{vol}_3(J'')^{\frac{3}{2}} \leq (1-\varepsilon)^{-18}\, l_{\mathrm{kor}}\lambda(q)^{\frac{1}{2}} s(J)^{\frac{3}{2}}.
\end{aligned}
$$

By a similar argument, but now using (3.5) from the statement of Lemma 3.2, we get that

$$\lim_{n\to\infty}\sqrt{n}\, v(\Omega;\mathcal{P}_n(J\subset H;\, f)) \geq (1-\varepsilon)^{30}\, l_{\mathrm{kor}}\lambda(q)^{\frac{1}{2}} s(J)^{\frac{3}{2}}.$$

Therefore, for large enough n, we get the desired estimates. □

6 Proof of Theorem 1.1

Proof of Theorem 1.1 Fix an $\varepsilon\in(0,1/3)$. There exists a tiling $\{L_j\}_{1\leq j\leq m}$ of $\partial\Omega$ consisting of Jordan measurable compact sets with non-empty interior such that

- for each $j=1,\ldots,m$, there is a $q_j\in L_j$ for which $L_j\subset U_{q_j,\varepsilon}$, where the latter comes from Lemma 5.6;
- $(1-\varepsilon)\lambda(q)\leq\lambda(q_j)\leq(1-\varepsilon)^{-1}\lambda(q)$, for all $q\in L_j$.

Then, recalling that $\lambda(q)=\dfrac{4M(\rho)(q)}{||\nabla\rho(q)||^3}$, we obtain estimates as follows:

$$4^{-\frac{1}{3}}\int_{\partial\Omega}\sigma_\Omega = \int_{\partial\Omega} 4^{\frac{1}{3}} M(\rho)(q)^{\frac{1}{3}}\frac{ds(q)}{||\nabla\rho(q)||} = \sum_{j=1}^m\int_{L_j}\lambda(q_j)^{\frac{1}{3}} ds(q_j)
\begin{cases} \leq (1-\varepsilon)^{-1}\displaystyle\sum_{j=1}^m \lambda(q_j)^{\frac{1}{3}} s(L_j)\\ \geq (1-\varepsilon)\displaystyle\sum_{j=1}^m \lambda(q_j)^{\frac{1}{3}} s(L_j).\end{cases} \tag{6.1}$$

Next, for all $j=1,\ldots,m$, we choose compact Jordan measurable sets J_j and H_j such that $J_j\subset \mathrm{int}_{\partial\Omega}L_j\subset\mathrm{int}_{\partial\Omega}H_j\subset U_{q_j,\varepsilon}$ and

$$s(J_j)\geq(1-\varepsilon)s(L_j). \tag{6.2}$$

1. We first estimate $v(\Omega;\mathcal{P}_n(f))$ from above. For $j=1,\ldots,m$, choose $P^j\in\mathcal{P}_{n_j}(L_j\subset H_j;\, f)$ such that $\mathrm{vol}(\Omega\setminus P^j)\leq(1-\varepsilon)^{-1}v(\Omega;\mathcal{P}_{n_j}(L_j\subset H_j;\, f))$. Let

P denote the intersection of all these P^j's. Then, P is an f-polyhedron with at most $n_1 + \cdots + n_m$ facets. Thus, by Lemma 5.6, for sufficiently large $n_1, \ldots, n_m$,

$$\begin{aligned} \mathrm{vol}(\Omega \setminus P) &\leq (1-\varepsilon)^{-1} \sum_{j=1}^{m} v(\Omega; \mathcal{P}_{n_j}(L_j \subset H_j; f)) \\ &\leq (1-\varepsilon)^{-20}\, l_{\mathrm{kor}} \sum_{j=1}^{m} \frac{\lambda(q_j)^{\frac{1}{2}} s(L_j)^{\frac{3}{2}}}{\sqrt{n_j}} \\ &= (1-\varepsilon)^{-20}\, l_{\mathrm{kor}} \sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{3}} s(L_j) \left(\frac{\lambda(q_j)^{\frac{1}{3}} s(L_j)}{n_j} \right)^{\frac{1}{2}}. \end{aligned} \tag{6.3}$$

Now, fix an $n \in \mathbb{N}_+$. Suppose we set

$$n_j = \left\lfloor \frac{\lambda(q_j)^{\frac{1}{3}} s(L_j)}{\sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{3}} s(L_j)} n \right\rfloor, \quad j = 1, \ldots, m. \tag{6.4}$$

Then,

$$n_1 + \cdots + n_m \leq n; \tag{6.5}$$

and

$$(1-\varepsilon) \frac{\lambda(q_j)^{\frac{1}{3}} s(L_j)}{\sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{3}} s(L_j)} n \leq n_j. \tag{6.6}$$

We use (6.5), substitute (6.6) in (6.3) and invoke (6.1) to get

$$\begin{aligned} v(\Omega; \mathcal{P}_n(f)) &\leq (1-\varepsilon)^{-21}\, l_{\mathrm{kor}} \left(\sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{3}} s(L_j) \right)^{\frac{3}{2}} \frac{1}{\sqrt{n}} \\ &\leq (1-\varepsilon)^{-24} \frac{l_{\mathrm{kor}}}{2} \left(\int_{\partial\Omega} \sigma_\Omega \right)^{\frac{3}{2}} \frac{1}{\sqrt{n}}, \end{aligned} \tag{6.7}$$

for n sufficiently large.

2. Next, we produce a lower bound for $v(\Omega; \mathcal{P}_n(f))$. For this, we first extend the tiling $\{L_j\}_{1 \leq j \leq m}$ of $\partial\Omega$ to a thin tubular neighborhood of $\partial\Omega$ in $\overline{\Omega}$, denoting the tile corresponding to L_j by $\hat{L}_j$. This can be done, for instance, by flowing each tile along the inward normal vector field for a short interval of time. Choose a $P_n \in \mathcal{P}_n(f)$ such that $\mathrm{vol}(\Omega \setminus P_n) \leq (1-\varepsilon)^{-1} v(\Omega; \mathcal{P}_n(f))$. Let n_j be the number of cuts of P_n that cover J_j. Due to the upper bound obtained in (6.7), $\lim_{n\to\infty} v(\Omega; \mathcal{P}_n(f)) = 0$. Thus, by Lemma 3.1, $\lim_{n\to\infty} \delta(P_n) = 0$. This permits us to choose n sufficiently large so that

- The n_j cuts that cover J_j lie in $\widehat{L}_j$.
- Each n_j is large enough so that the bounds in Lemma 5.6 hold.

Thus, invoking Lemma 5.6 and using (6.2), we have that

$$\begin{aligned}\mathrm{vol}(\Omega \setminus P_n) &\geq \sum_{j=1}^{m} \mathrm{vol}\left(\widehat{L}_j \setminus P_n\right) \geq \sum_{j=1}^{m} v(\Omega; \mathcal{P}_{n_j}(J_j \subset L_j; f)) \\ &\geq (1-\varepsilon)^{31}\, l_{\mathrm{kor}} \sum_{j=1}^{m} \frac{\lambda(q_j)^{\frac{1}{2}} s(J_j)^{\frac{3}{2}}}{\sqrt{n_j}} \\ &\geq (1-\varepsilon)^{33}\, l_{\mathrm{kor}} \sum_{j=1}^{m} \frac{\lambda(q_j)^{\frac{1}{2}} s(L_j)^{\frac{3}{2}}}{\sqrt{n_j}}.\end{aligned}$$

Now, Hölder's inequality gives

$$\sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{3}} s(L_j) = \sum_{j=1}^{m} \left(\frac{\lambda(q_j) s(L_j)^3}{n_j}\right)^{\frac{1}{3}} n_j^{\frac{1}{3}} \leq \left(\sum_{j=1}^{m} \frac{\lambda(q_j)^{\frac{1}{2}} s(L_j)^{\frac{3}{2}}}{\sqrt{n_j}}\right)^{\frac{2}{3}} \left(\sum_{j=1}^{m} n_j\right)^{\frac{1}{3}}$$

Thus, using one of the estimates in (6.1),

$$\begin{aligned}\mathrm{vol}(\Omega \setminus P_n) &\geq (1-\varepsilon)^{33}\, l_{\mathrm{kor}} \left(\sum_{j=1}^{m} \lambda(q_j)^{\frac{1}{3}} s(L_j)\right)^{\frac{3}{2}} \frac{1}{\sqrt{n_1 + \cdots + n_m}} \\ &\geq (1-\varepsilon)^{35}\, l_{\mathrm{kor}} \left(\frac{1}{4^{1/3}} \int_{\partial\Omega} \sigma_\Omega\right)^{\frac{3}{2}} \frac{1}{\sqrt{n}}.\end{aligned}$$

By our choice of P_n,

$$v(\Omega; \mathcal{P}_n(f)) \geq (1-\varepsilon)^{36} \frac{l_{\mathrm{kor}}}{2} \left(\int_{\partial\Omega} \sigma_\Omega\right)^{\frac{3}{2}} \frac{1}{\sqrt{n}}, \tag{6.8}$$

for all n sufficiently large.

Finally, we combine (6.8) and (6.7), and recall that $\varepsilon \in (0, 1/3)$ was arbitrary, to declare the proof of Theorem 1.1 complete. □

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Appendix: Power Diagrams in the Heisenberg Group

The Euclidean Plane

Let $D(a; r) \subset \mathbb{R}^2$ be a disk of radius r centered at $a \in \mathbb{R}^2$. The *power* of a point $z = (x, y) \in \mathbb{R}^2$ with respect to $D = D(a; r)$ is the number

$$\mathrm{pow}(z, D) = |z - a|^2 - r^2.$$

Note that if z is outside the disk D, then $\mathrm{pow}(z, D)$ is the square of the length of a line segment from z to a point of tangency with ∂D. Thus, it is a generalized distance between z and ∂D. For a collection $\mathscr{D}$ of disks in the plane, the *power diagram* or *Laguerre–Dirichlet–Voronoi tiling* of $\mathscr{D}$ is the collection of all

$$\mathrm{cell}(D) = \{z \in \mathbb{R}^2 : \mathrm{pow}(z, D) < \mathrm{pow}(z, D^*), \quad \forall D^* \in \mathscr{D} \setminus \{D\}\}, \; D \in \mathscr{D}.$$

If $\mathscr{D}$ consists of equiradial disks, the power diagram reduces to the Dirichlet–Voronoi diagram of the centers of the disks. In general, the power diagram of any $\mathscr{D}$ gives a convex tiling of the plane (Fig. 2).

Power diagrams occur naturally and have found several applications (see [2], for instance). From the point of view of polyhedral approximations, power diagrams (in $\mathbb{R}^{d-1}$) are intimately related to the constant ldiv_{d-1} in (1.2) (see [15] and [7] for explicit details).

The Heisenberg Group

Let $K(0; \delta) = \{z' \in \mathbb{H} : |z_1|^4 + (x_2)^2 < \delta^4\}$ be a Korányi sphere in $\mathbb{H}$ (see (4.1)). We define the *horizontal power* of a point $z' \in \mathbb{H}$ with respect to $K = K(0; \delta)$ as

$$\mathrm{hpow}(z', K) = \begin{cases} |z_1|^2 - \sqrt{\delta^4 - (x_2)^2}, & \text{if } |x_2| \le \delta^2; \\ \infty, & \text{otherwise.} \end{cases}$$

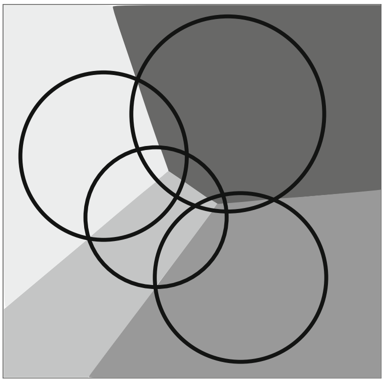

Fig. 2 A power diagram in the plane

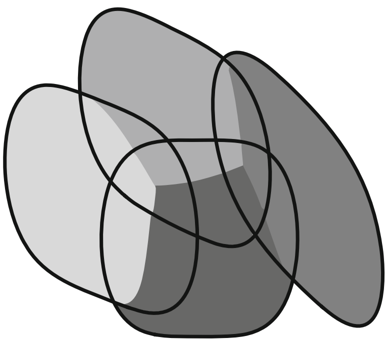

Fig. 3 A $\{x_1 = 0\}$-slice of a horizontal power diagram in $\mathbb{H}$

Note that $K_c := K \cap \{x_2 = c\}$ is a (possibly empty) disk in the $\{x_2 = c\}$ plane, and $\mathrm{hpow}((z_1, x_2), K) = \mathrm{pow}(z_1, K_{x_2})$, where the right-hand side—being a generalized distance—is set as ∞ when K_{x_2} is empty. hpow is then extended to all Korányi spheres to be left-invariant under $\cdot_{\mathbb{H}}$ (defined in Sect. 4). For a collection $\mathscr{K}$ of Korányi spheres in $\mathbb{H}$, define the *horizontal power diagram* or *Laguerre–Korányi tiling* of $\mathscr{K}$ to be the collection of all

$$\mathrm{hcell}(K) = \left\{ z' \in \bigcup_{K \in \mathscr{K}} K : \mathrm{hpow}(z', K) < \mathrm{hpow}(z', K^*), \quad \forall K^* \in \mathscr{K} \setminus \{K\} \right\}, K \in \mathscr{K}.$$

Then, $\mathrm{hcell}(K) \subset K$, for all $K \in \mathscr{K}$ (Fig. 3).

We now give two reasons why this concept is useful for us. Let

$$\begin{aligned} \mathrm{dil}_\xi &: (z_1, x_2) \mapsto (\xi z_1, \xi^2 x_2), \\ \mathrm{dil}_{w',\xi} &: z' \mapsto w' \cdot_{\mathbb{H}} \mathrm{dil}_\xi(-w' \cdot_{\mathbb{H}} z') \end{aligned}$$

be the dilations in $\mathbb{H}$ centered at the origin and w', respectively. Then,

(1) $\mathrm{dil}_{w',\xi}(K(w', \delta)) = K(w', \xi\delta)$,
(2) $\mathrm{hpow}(\mathrm{dil}_{w',\xi}(z'), K(w', \delta)) = \xi^2\,\mathrm{hpow}(z', K(w', \xi^{-1}\delta))$, and
(3) if $\mathscr{K} = \{K_j := K(a_j, \delta_j) : j = 1, \ldots, m\}$, then, $\mathrm{dil}_{a_j,\xi}\,\mathrm{hcell}\left(K_l\right) \cap \mathrm{dil}_{a_k,\xi}\,\mathrm{hcell}\left(K_j\right) = \emptyset$, for all $1 \leq l < j \leq m$ and $\xi \leq 1$.

Now, consider the Siegel domain $\mathcal{S}$ and the function $f_{\mathcal{S}}$ studied in Sect. 4. The cuts of any $f_{\mathcal{S}}$-polyhedron P over $J \subset \partial S$ project to a collection $\mathscr{K}_P$ of Korányi balls in $\mathbb{C} \times \mathbb{R}$ that form a covering of J'. The (open) facets of P project to the horizontal power diagram of $\mathscr{K}_P$. This perspective facilitates the proof of

Lemma 6.1 *The cuts of $f_{\mathcal{S}_\lambda}$, $\lambda > 0$, are Jordan measurable and satisfy the doubling property (3.3) for any $\delta_{f_{\mathcal{S}_\lambda}} > 0$ and $D(t) = (1+t)^3$.*

Proof The Jordan measurability of the cuts is obvious. Now, without loss of generality, we may assume $\lambda = 1$ (the map $(z, w) \mapsto (\lambda z, \lambda w)$ can be used to handle the other cases). Let $H \subset \partial\mathcal{S}$ be a compact set, $\{w^j\}_{1\le j\le m} \subset H$, $\{\delta_j\}_{1\le j\le m} \subset (0,\infty)$ and $t > 0$. For $j = 1,\dots,m$, let

$$C_j(t) := C(w_j, (1+t)\delta_j; f_{\mathcal{S}}),$$
$$v^j = (w^j)' = (w_1^j, u_2^j),$$

and (see (4.1))

$$K_j(t) := C_j(t)' = K\left(v^j; \sqrt{(1+t)\delta_j}\right).$$

Consider $\mathscr{K} = \{K_j(t) : 1 \le j \le m\}$ and the corresponding horizontal power diagram $\{\mathrm{hcell}_j(t) = \mathrm{hcell}(K_j(t)) : 1 \le j \le m\}$. Then, setting $dz' = dx_1 dy_1 dx_2$, we have, by a change of variables and (1), (2) and (3) above, that

$$\begin{aligned}
&\mathrm{vol}\left(\bigcup_{j=1}^m C_j(t)\right)\\
&= \int_{\cup_{j=1}^m K_j(t)} \max_{1\le j\le m}\left\{\mathrm{Re}\sqrt{\delta_j^2 - (x_2 - u_2^j + 2\,\mathrm{Im}\, z_1\overline{w_1}^j)} - |z_1 - w_1^j|^2\right\} dz'\\
&= \int_{\cup_{j=1}^m K_j(t)} \max_{1\le j\le m}\{-\mathrm{hpow}(z', K_j(t))\} dz'\\
&= -\sum_{j=1}^m \int_{\mathrm{hcell}_j(t)} \mathrm{hpow}(z', K_j(t)) dz'\\
&= -(1+t)^2 \sum_{j=1}^m \int_{\mathrm{dil}_{v^j,\frac{1}{\sqrt{1+t}}}(\mathrm{hcell}_j(t))} \mathrm{hpow}\left(\mathrm{dil}_{v^j,\sqrt{1+t}}(\zeta), K_j(t)\right) d\zeta\\
&= -(1+t)^3 \sum_{j=1}^m \int_{\mathrm{dil}_{v^j,\frac{1}{\sqrt{1+t}}}(\mathrm{hcell}_j(t))} \mathrm{hpow}\left(\zeta, K_j(0)\right) d\zeta\\
&\le (1+t)^3 \int_{\cup_{j=1}^m K_j(0)} \max\left\{-\mathrm{hpow}\left(\zeta, K_j(0)\right) : 1 \le j \le m\right\} d\zeta\\
&= (1+t)^3\, \mathrm{vol}\left(\bigcup_{j=1}^m C_j(0)\right), \quad \forall t \ge 0.
\end{aligned}$$

□

The computations in the above proof also show that

$$l_{\mathrm{kor}} = \lim_{n\to\infty} \sqrt{n} \inf\left\{-\sum_{K\in\mathscr{K}} \int_{\mathrm{hcell}(K)} \mathrm{hpow}(z', K) dz' : I \subset \bigcup_{K\in\mathscr{K}} K,\ \#(\mathscr{K}) \le n\right\},$$

where I is the unit square in $\mathbb{C} \times \mathbb{R}$ (see Sect. 4). Our proof of Lemma 4.1 yields bounds for l_{kor} as follows:

$$0.0003 \approx \frac{4\sqrt{2}}{\pi^2 3^7} \leq l_{\mathrm{kor}} \leq \frac{5\sqrt{5}\pi}{3\sqrt{2}} \approx 8.2788.$$

It would be interesting to know if computations, similar to the ones carried out by Böröczky and Ludwig in [7] for ldiv_2, can be done to find the exact value of l_{kor}.

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