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STABILITY OF MULTICLUSTER SYNCHRONIZATION

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A system of many coupled oscillators on a network can show multicluster synchronization. We obtain existence conditions and stability bounds for such a multicluster synchronization. When the oscillators are identical, we obtain the interesting result that network structure alone can cause multicluster synchronization to emerge even when all the other parameters are the same. We also study occurrence of multicluster synchronization when two different types of oscillators are coupled.

Keywords: Synchronization; networks; clusters; stability.

1. Introduction

Synchronized behavior of a network of interacting dynamical systems or oscillators is a commonly observed phenomena [Kuramoto, 1984; Pikovsky *et al.*, 2001; Arenas *et al.*, 2008]. Synchronous behavior is observed when all the systems behave in the same fashion, i.e. they all show the same properties at the same time. These studies assume some form of coupling, and investigate properties such as stability of the synchronized state, its bifurcations, etc. [Pecora & Carroll, 1998; Barahona & Pecora, 2002; Nishikawa *et al.*, 2003; Timme & Geisel, 2004; Belykh *et al.*, 2005; Zhou *et al.*, 2006; Oh *et al.*, 2005; Amritkar & Rangarajan, 2006; Boccaletti *et al.*, 2002]. Most of these studies concentrate on the synchronous behavior of the entire network. However, in nature, one finds many examples where the entire network may not show a synchronized behavior but may split into several clusters of independent synchronized or organized behavior. Several recent studies have found evidence of multicluster synchronization in asymptotic state [Jalan & Amritkar, 2003; Jalan *et al.*, 2005; Amritkar & Jalan, 2005; Amritkar *et al.*, 2005], in transient state [Arenas *et al.*, 2006], in modular structures [Angelini *et al.*, 2006] or in hierarchical synchronization where only a part of the network synchronizes [Zhou *et al.*, 2006; Zhou & Kurths, 2006]. Despite these recent efforts, the relation between the network topology and multicluster synchronization and stability of such synchronization is not clear. In this letter, we address these issues.

There are two possible ways in which multicluster synchronization in a network of interacting dynamical systems may be observed. First, the structure of the network may be such that it introduces different clusters in a natural way. As we will demonstrate later, even in similarly coupled identical oscillators we can get multicluster synchronization solely due to the network properties. Second, there may be different groups of nodes which have different dynamics either because the form of the equations is different or the parameters are different.

In this paper, we include both the above possibilities in our formalism of multicluster synchronization. First, we review the conditions for stability of single cluster synchronization. The conditions can be expressed in terms of a master stability function so that it is applicable to different kinds of networks [Pecora & Carroll, 1998]. We then analyze the conditions for the occurrence of multicluster synchronization. Using linear stability analysis, we obtain stability conditions for this state. For a special class of networks, "separable networks," it is possible to cast these conditions in a master stability formalism. We obtain a master stability function for each cluster. For a general class of networks, the master stability formalism can provide a lower bound for the coupling parameters. Next, we consider examples of a variety of networks where we demonstrate multicluster synchronization for both identical and different oscillators. For the separable class of networks, the master stability functions can be used to obtain the stability range. For a general class of networks, we show that the stability of multicluster synchronization is consistent with the lower bound obtained earlier.

The condition for the stability of synchronized state that we use in this paper is based on the largest transverse Lyapunov exponent and is also the condition used in the master stability function [Pecora & Carroll, 1998]. It is a weak condition. If there are unstable invariant sets in the synchronized state [Ashwin *et al.*, 1994] or locally unstable areas on the attractor [Gauthier & Bienfang, 1996; Rulkov & Sushchik, 1997], then in the presence of noise or parameter mismatch, it can cause bubbling and bursting of the system away from the synchronized state. However, as noted in [Pecora & Carroll, 1998] the criterion based on the largest Lyapunov exponent is universally used. It gives reasonably good numerical as well as analytical results in most cases. It is in this sense that we use our stability criterion with the proviso that in some cases bubbling and bursting may occur in the presence of noise or parameter mismatch.

2. Single Cluster Synchronization

Let us first consider the conditions for the occurrence and stability of the single cluster synchronization where all the nodes synchronize together. Even though this problem has already been analyzed [Pecora & Carroll, 1998], we include a brief description since our treatment incorporates the general condition for the existence of the synchronized state. Moreover, this generalization sets the stage for analyzing multicluster synchronization. We denote an n-cluster synchronization by nCS and thus a single cluster synchronization by 1CS. Consider a network of N nodes of interacting dynamical systems or oscillators. Let $\mathbf{x}^i(t) \in R^m$ be the m-dimensional variable of the ith node. Let the uncoupled dynamics of each node be defined by the function $\mathbf{f}(\mathbf{x}^i(t))$ and the coupling by the function $\mathbf{u} : R^m \to R^m$. Let G be the $N \times N$ coupling matrix of the network. We allow the possibility of directed networks and also links with different weights. The form of dynamical systems suitable for such studies is discussed in [Pecora & Carroll, 1998]. Thus, the dynamics of ith node is given by

$$\dot{\mathbf{x}}^i(t) = \mathbf{f}(\mathbf{x}^i(t)) + \sum_j G_{ij}\mathbf{u}(\mathbf{x}^j(t)). \tag{1}$$

A single cluster synchronization (1CS) is defined by $\mathbf{x}^1 = \cdots = \mathbf{x}^N = \mathbf{x}$. The 1CS is a solution of Eq. (1) provided the coupling matrix satisfies the condition that

$$\sum_j G_{ij} = g, \quad \forall\, i, \tag{2}$$

where g is a constant independent of i. This condition is a generalization of the synchronization condition which is normally used. We note that this generalization is not very important for 1CS, since by suitably redefining the local dynamics we can set $g = 0$. However, for nCS it becomes important. The synchronized state is a solution of

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + g\mathbf{u}(\mathbf{x}). \tag{3}$$

If $g = 0$ then the synchronized state is a solution of the uncoupled dynamics, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$.

The condition (2) ensures that G has one eigenvector of the type $e_1^R = (1, \ldots, 1)^T$ with eigenvalue $\gamma_1 = g$. This eigenvector defines the synchronization manifold and it has the dimension m. All the remaining eigenvectors belong to the transverse manifold. Accordingly, the Lyapunov exponents can be separated into two sets, corresponding to the synchronization and transverse manifolds. The 1CS is stable provided all the transverse Lyapunov exponents are negative.

We now analyze the linear stability of the synchronized state. Linearizing each oscillator (1) about the synchronized trajectory $\mathbf{x}$ gives [Rangarajan & Ding, 2002; Chen *et al.*, 2003]

$$\dot{Z} = D\mathbf{f}Z + D\mathbf{u}ZG^T, \tag{4}$$

where $Z = (\mathbf{z}^1, \mathbf{z}^2, \ldots, \mathbf{z}^N)$, $\mathbf{z}^i = \mathbf{x}^i - \mathbf{x}$, and both $D\mathbf{f}$, $D\mathbf{u}$ are evaluated at $\mathbf{x}$.

Let $\gamma_k, e_k^L, k = 1, \ldots, N$ be the eigenvalues and eigenvectors of G^T. Acting Eq. (4) on e_k^L we get

$$\dot{Z}e_k^L = D\mathbf{f}Ze_k^L + D\mathbf{u}Z\gamma_k e_k^L. \tag{5}$$

Let us define m-dimensional vectors $\phi_k = Ze_k^L$. Then, Eq. (5) reduces to

$$\dot{\phi}_k = [D\mathbf{f} + \gamma_k D\mathbf{u}]\phi_k, \tag{6}$$

where $k = 1, \ldots, N$.

Since γ_k can be complex, treating it as a complex parameter α, we can construct the master stability equation as [Pecora & Carroll, 1998]

$$\dot{\phi} = [D\mathbf{f} + \alpha D\mathbf{u}]\phi. \tag{7}$$

When $g \neq 0$, the synchronized state depends on g [Eq. (3)] and consequently so does the master stability equation. We can determine the master stability function $\lambda_{\max}$, which is the largest Lyapunov exponent for Eq. (7), as a surface over the complex plane defined by α. Each g value will give rise to a different master stability function. The 1CS is stable if the master stability function is negative at each of the eigenvalues γ_k $(k \neq 1)$.

3. Two-Cluster Synchronization

Here, we will obtain the conditions for the existence of the 2CS and analyze the stability of the same. The results can be easily extended to the n-cluster synchronization (nCS). In general, we assume that the nodes of the two clusters are governed by different dynamical systems and denote the variables by $\mathbf{x}^i, i = 1, \ldots, N_1$ and $\mathbf{y}^j, j = 1, \ldots, N_2$ of dimension m and N_1 and $N_2 = N - N_1$ are the number of nodes of the two clusters. We note that it is possible to consider the case when the two systems have different dimensions. A simple way to do this in the present analysis is to pad up the lower dimensional system with zeros.

The dynamics can be written as

$$\begin{aligned}\dot{\mathbf{x}}^i &= \mathbf{f}(\mathbf{x}^i) + \sum_{l=1}^{N_1} A_{il}\mathbf{u}(\mathbf{x}^l) + \sum_{n=1}^{N_2} B_{in}\mathbf{v}(\mathbf{y}^n),\\ \dot{\mathbf{y}}^j &= \mathbf{g}(\mathbf{y}^j) + \sum_{l=1}^{N_1} C_{jl}\mathbf{u}(\mathbf{x}^l) + \sum_{n=1}^{N_2} D_{jn}\mathbf{v}(\mathbf{y}^n),\end{aligned} \tag{8}$$

where the coupling matrix G is split into four blocks A, B, C, D (see Fig. 1). The different functions and matrices in this equation have appropriate dimensions. For the sake of simplicity many of the dimensions here as well as in subsequent expressions are not stated explicitly and can be deduced from the context.

We define the 2CS state by $\mathbf{x}^1 = \cdots = \mathbf{x}^{N_1} = \mathbf{x}$ and $\mathbf{y}^1 = \cdots = \mathbf{y}^{N_2} = \mathbf{y}$. Existence of the 2CS requires that it be a solution of the dynamics [Eq. (8)]. This implies that G should have eigenvectors of the form $e^R = (\mu, \ldots, \mu, \nu, \ldots, \nu)^T$. There will be two such linearly independent eigenvectors and they lead to the synchronization manifold. Using these considerations, it is easy to show that

$$\begin{aligned}\sum_j A_{ij} = a, \quad \sum_j B_{ij} = b,\\ \sum_j C_{ij} = c, \quad \sum_j D_{ij} = d, \quad \forall i,\end{aligned} \tag{9}$$

where a, b, c, d are constants. The condition (2) for the existence of the 1CS will be satisfied if $a + b = c + d$. We note that the synchronization manifold for the 2CS has dimension $2m$ while the transverse manifold has dimension $(N-2)m$.

Using Eqs. (8) and (9), the synchronized variables of the 2CS are seen to satisfy the equations

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + a\mathbf{u}(\mathbf{x}) + b\mathbf{v}(\mathbf{y}),\\ \dot{\mathbf{y}} &= \mathbf{g}(\mathbf{y}) + c\mathbf{u}(\mathbf{x}) + d\mathbf{v}(\mathbf{y}).\end{aligned} \tag{10}$$

Let us now consider the conditions for the stability of the 2CS. Linearizing each oscillator [Eq. (8)]

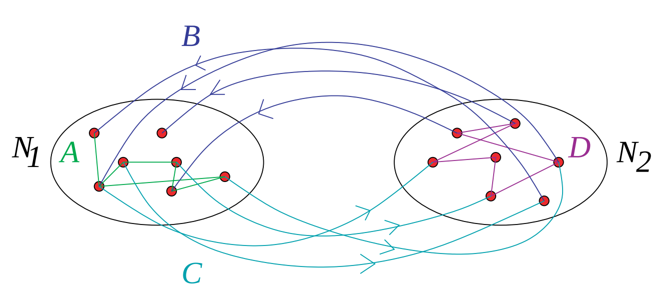

Fig. 1. Schematic representation of the two clusters and the splitting of the coupling matrix into four blocks, A, B, C, D.

about the 2CS trajectory gives

$$(\dot{Z}^1, \dot{Z}^2) = (D\mathbf{f}Z^1, D\mathbf{g}Z^2) + (D\mathbf{u}Z^1, D\mathbf{v}Z^2)G^T, \tag{11}$$

where $\mathbf{Z}^1 = (\mathbf{x}^1 - \mathbf{x}, \ldots, \mathbf{x}^{N_1} - \mathbf{x})$, $\mathbf{Z}^2 = (\mathbf{y}^1 - \mathbf{y}, \ldots, \mathbf{y}^{N_2} - \mathbf{y})$, and $D\mathbf{f}$, $D\mathbf{u}$ and $D\mathbf{g}$, $D\mathbf{v}$ are respectively evaluated at $\mathbf{x}$ and $\mathbf{y}$. Using the eigenvectors and eigenvalues of G^T, Eq. (11) can be written as

$$\begin{aligned}(\dot{Z}^1, \dot{Z}^2)e_k^L = (D\mathbf{f}Z^1, D\mathbf{g}Z^2)e_k^L \\ + (D\mathbf{u}Z^1, D\mathbf{v}Z^2)\gamma_k e_k^L.\end{aligned} \tag{12}$$

The left eigenvectors associated with the two right eigenvectors of the form $e^R = (\mu, \ldots, \mu, \nu, \ldots, \nu)^T$ correspond to the synchronization manifold and we denote them by e_1^L and e_2^L. The corresponding eigenvalues are given by

$$\gamma_{1,2} = \frac{a + d \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{2}. \tag{13}$$

The remaining vectors (e_k^L, $k = 3, \ldots, N$) correspond to the transverse manifold. For the 2CS to be stable the transverse Lyapunov exponents must be all negative.

In principle, one would follow the same procedure that we adopted for the 1CS case and obtain the master stability function. It is, however, difficult to obtain the master stability function for the 2CS state directly. We therefore employ the following procedure. Returning to Eq. (12) we write the eigenvector e_k^L in the form $e_k^L = (e_{k1}^T, e_{k2}^T)^T$. Thus, Eq. (12) can be written as

$$\begin{aligned}\dot{Z}^1 e^1 + \dot{Z}^2 e^2 = [D\mathbf{f} + \gamma_k D\mathbf{u}]e^1 \\ + [D\mathbf{g} + \gamma_k D\mathbf{v}]e^2.\end{aligned} \tag{14}$$

We define m-dimensional vectors ϕ_k and ψ_k as

$$\begin{aligned}p_{k1}\phi_k = Z^1 e_{k1}, \\ p_{k2}\psi_k = Z^2 e_{k2},\end{aligned} \tag{15}$$

where the weights $p_{ki}^2 = (e_{ki})^\dagger e_{ki}, i = 1, 2$ with $p_{k1}^2 + p_{k2}^2 = 1$. The weights p_{ki} are introduced for book keeping (see also Appendix A). In terms of ϕ_k and ψ_k, Eq. (12) becomes

$$\begin{aligned}p_{k1}\dot{\phi}_k + p_{k2}\dot{\psi}_k = [D\mathbf{f} + \gamma_k D\mathbf{u}]p_{k1}\phi_k \\ + [D\mathbf{g} + \gamma_k D\mathbf{v}]p_{k2}\psi_k.\end{aligned} \tag{16}$$

Following the procedure used for 1CS, we can now construct the master stability equation for the 2CS as

$$\begin{aligned}p_1\dot{\phi} + p_2\dot{\psi} = [D\mathbf{f} + \alpha D\mathbf{u}]\, p_1\phi \\ + [D\mathbf{g} + \alpha D\mathbf{v}]\, p_2\psi,\end{aligned} \tag{17}$$

where α is a complex parameter and $0 \le p_1, p_2 \le 1$ are real parameters. These are m equations that need to be solved using the 2CS solution obtained from Eqs. (10).

We first consider a special class of networks, referred to as a *separable class*, for which Eq. (17) can be solved exactly. These are networks for which the transverse eigenvectors split into two independent subspaces corresponding to the two clusters, i.e. they are of the form

$$(\mu_1, \ldots, \mu_{N_1}, 0, \ldots, 0)^T \quad \text{or} \quad (0, \ldots, 0, \nu_1, \ldots, \nu_{N_2})^T$$

The complete bipartite network is an example [Amritkar *et al.*, 2005]. For this separable class of networks, Eq. (17) can be written as a set of two equations, corresponding to the two clusters, and are obtained by putting $p_2 = 0$ and $p_1 = 0$, respectively.

$$\begin{aligned}\dot{\phi} = [D\mathbf{f} + \alpha D\mathbf{u}]\phi, \\ \dot{\psi} = [D\mathbf{g} + \alpha D\mathbf{v}]\psi.\end{aligned} \tag{18}$$

These are $2m$ equations and can be solved using the 2CS solution obtained from Eqs. (10). Thus, we can determine the cluster master stability functions (CMSFs) which are given by the largest Lyapunov exponents for the two equations in (18), as two surfaces over the complex plane defined by α. We note that the CMSF depend on the parameters a, b, c, d which determine the evolution of 2CS state variables $\mathbf{x}$ and $\mathbf{y}$. The stability condition for 2CS is that at the transverse eigenvalues the CMSF for the respective clusters must be negative [Amritkar *et al.*, 2005].

We note that Eqs. (18) can also be obtained from Eq. (11) by making the variations of $\mathbf{y}$ ($\mathbf{x}$) zero (that is, make Z^2 (Z^1) zero). Hence, the Lyapunov exponents obtained from Eqs. (18) are nothing but the conditional Lyapunov exponents for $\mathbf{x}$ and $\mathbf{y}$, respectively. In this case, the parameter α will correspond to the eigenvalues of the matrices A and D in the equations (18).

For a general class of networks, it is not possible to solve Eq. (17) for ϕ and ψ. In general, the CMSF can serve as a lower bound (or a sufficiency condition) for the stability of 2CS so that if for all the transverse eigenvalues $\alpha = \gamma_k$ both the CMSF are negative then the 2CS is stable. However, we have not been able to obtain a higher bound for the stability. The reason for the lower bound is that the solution of Eq. (17) has both clusters, involving the projection of the maximum Lyapunov exponent for each cluster along some other direction, thereby reducing the value of the maximum Lyapunov exponent.

A solution of Eq. (17) can be obtained using Moore–Penrose pseudo-inverse [Kaipo & Somersalo, 2005] and is given by (see Appendix B)

$$\begin{pmatrix} \dot{\phi} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} p_1^2 D_\phi & p_1 p_2 D_\psi \\ p_1 p_2 D_\phi & p_2^2 D_\psi \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}. \quad (19)$$

As discussed in Appendix B, this may be used as an approximate solution of Eq. (17).

For $m = 1$, the pseudo-inverse solution can be solved and the master stability function (MSF) (or the maximum Lyapunov exponent for the entire network) can be expressed as [see Eq. (B.3)]

$$\lambda_{\max} = p_1^2 \lambda_\phi + p_2^2 \lambda_\psi, \quad (20)$$

where λ_ϕ and λ_ψ are the CMSF for the two clusters obtained from $\langle D_\phi \rangle$ and $\langle D_\psi \rangle$, i.e. Eqs. (18). Unfortunately, for $m > 1$, Eq. (20) is not valid.

The above formalism can be easily generalized for nCS. The number of conditions that the coupling matrix must satisfy become n^2 [see Eqs. (2) and (9)]. The synchronization manifold has dimension nm and the transverse manifold $(N-n)m$. The synchronized variables satisfy a set of nm equations similar to Eqs. (3) and (10). The master stability equation is similar to Eqs. (7) and (17). For the separable case, we have a set of n equations similar to Eqs. (18).

We note that our analysis can help in designing networks that give multicluster synchronization. The necessary condition (though not sufficient) is that all the transverse eigenvectors of the coupling matrix satisfy the relations of the type given by Eqs. (A.4). In addition, if the transverse eigenvectors belong to the separable class of networks introduced above then no further condition on the network structure is required. That part of the condition from the dynamics about the transverse Lyapunov exponents is distinct from the network structure and must be satisfied independently.

4. Examples

We now consider some examples.

4.1. *Identical Rössler systems*

When identical oscillators are coupled, we would naively expect that only a single synchronized cluster would be possible. We demonstrate below the interesting result that one can get two-cluster synchronization even in this case. Since all system parameters are identical, the 2CS state is purely a result of network properties.

We consider symmetrically coupled, identical Rössler systems

$$\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{x}) = (-x_2 - x_3, x_1 + a_r x_2, b_r + x_3(x_1 - c_r))^T, \quad (21)$$

with $a_r = b_r = 0.2, c_r = 7.0$. We choose linear coupling functions as

$$\mathbf{u}(\mathbf{x}) = \mathbf{v}(\mathbf{x}) = (x_1, 0, 0)^T \quad (22)$$

Let $a+b = c+d = 0$ and $b = c$. Thus, both conditions (2) and (9) are satisfied. In this case, both 1CS and 2CS are possible and which synchronization actually emerges depends on the stability of the two states of synchronization. The above system (with symmetric coupling) shows 1CS if all the eigenvalues except $\gamma_1 = 0$ fall in a fixed range $[\alpha_1, \alpha_2]$ as determined from Eq. (7) for the master stability function of 1CS (see Fig. 1 in [Pecora & Carroll, 1998]). Now if we choose a such that the second eigenvalue $\gamma_2 = 2a$ [Eq. (13)] is greater than α_2, then 1CS is unstable. However, 2CS can be stable. The two CMSFs for the 2CS as determined

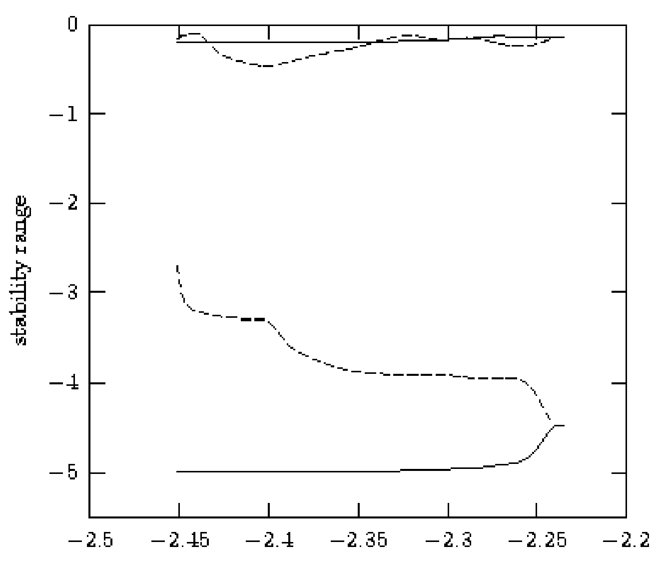

Fig. 2. The stability ranges for the two clusters are shown as a function of the parameter a for the 2CS of identical Rössler oscillators with $a + b = c + d = 0$. Two ranges are determined using Eqs. (18) and shown as solid and dashed lines, respectively.

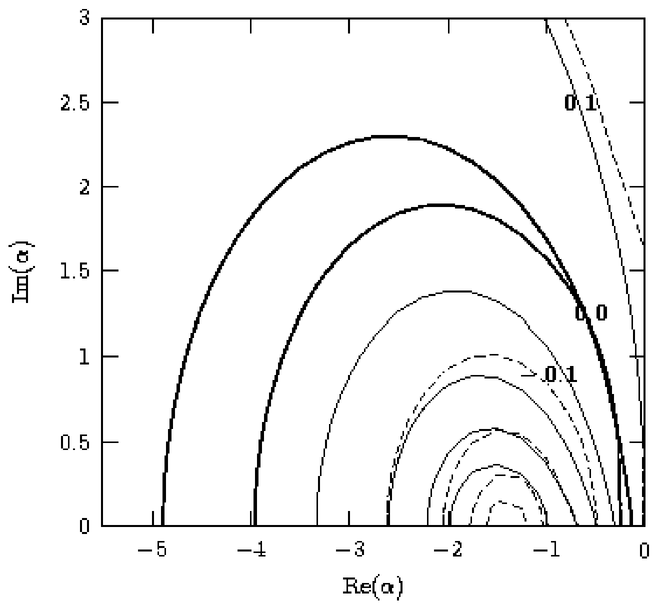

Fig. 3. The contour plots of the CMSFs are shown in the complex α plane for identically coupled Rössler oscillators with $a + b = c + d = 0$ and $a = d = -2.35$. There are two sets of contours corresponding to Eqs. (18). The bold solid contours are for the zero of CMSFs. The values are shown so as to correspond to both the clusters for $\lambda_{\max} = 0.1, 0.0, -0.1$. The values of the last three contours are not shown explicitly.

from Eqs. (18) depend only on a (with $b = c = -a$ and $d = a$). Figure 2 shows the plot of the stability ranges for the two clusters, as a function of a. There are two ranges corresponding to the two clusters and are obtained using Eqs. (18). For the separable networks, the 2CS is stable if the CMSFs corresponding to the transverse eigenvalues of each cluster are all negative. For a general network, the common area acts as a lower bound to the stability, i.e. if all the transverse eigenvalues fall in this area then 2CS is stable. Figure 3 shows the CMSFs in the complex α plane for $a = -2.35$.

We first demonstrate 2CS by coupling identical Rössler systems as described above using a complete bipartite network with $N = N_1 + N_2$. We take $A = aI_{N_1}, D = aI_{N_2}, B_{ij} = -a/N_1$ and $C_{ij} = -a/N_2$. We denote this as network I. The coupling matrix G has eigenvalues $\gamma_1 = 0$ and $\gamma_2 = 2a$ and the N_1+N_2-2 fold degenerate transverse eigenvalues $\gamma_k = a, k = 3, \ldots, N$ [Amritkar *et al.*, 2005]. This network belongs to the separable class as discussed in the previous section. Since all the transverse eigenvalues are a, the 2CS is stable if $\alpha = a$ falls within the stability range of both the clusters. We have numerically verified the result.

It is possible to construct several other forms of coupling matrices with the same stability ranges as in Fig. 2. We only need to ensure that $a + b = c + d = 0$ with $b = c$. First, let $A_{ij} = -s/N_1, i \neq j$ and $A_{ii} = a + s(N_1 - 1)/N_1$. Note that $\sum_j A_{ij} = a$ still holds. Taking the remaining matrices B, C, D to be the same as in the previous example, we obtain network II. The eigenvalues of G are $\gamma_1 = 0, \gamma_2 = 2a$, and an $N_1 - 1$ fold degenerate transverse eigenvalue $a+s$ and another N_2-1 fold degenerate transverse eigenvalue a. We can determine the stability bounds of the 2CS state using Fig. 2. As the eigenvalue $a+s$ corresponding to the first cluster, changes with s (the eigenvalue corresponding to the second cluster remains constant at a), 2CS remains stable if $a + s$ remains within the stability range of the first clusters. For example, when $a = -2.35$, 2CS is stable for $s \in [-2.16, 1.5][-2.29\ldots, 2.1\ldots]$ (see Fig. 2). We have verified these results numerically.

We can consider several other special types of networks. Instead, we consider a very general network where the coupling constants are randomly generated except that the two conditions $a + b = c+d = 0$ and $b = c$ are satisfied. We have considered many randomly generated networks of this type and they all obey the lower bound as described above.

We note that in all the examples considered above, the 2CS is obtained purely from the network properties.

Next, we considered identical Rössler systems as above, but now $a + b \neq c + d$. Here, we cannot have 1CS for any parameter values. We considered several networks with the same Rössler system parameters and found to obey the lower bound as described above.

4.2. *Rössler + Lorenz*

We choose the Rössler systems as above and Lorenz as

$$\mathbf{g}(y) = (\sigma(y_2 - y_1), y_1(\rho - y_3) - y_2, y_1y_2 - \beta y_3)^T \tag{23}$$

with $\sigma = 10, \rho = 28, \beta = 8/3$. The two CMSFs in the complex α plane for $a = -0.5, b = 0.25, c = 2.5, d = -2.5$ were computed and are shown in Fig. 4. We have chosen different networks and verified the stability properties of the 2CS. For network I, the transverse eigenvalues are a and d for the Rössler and Lorenz systems respectively and 2CS is stable. For network II discussed above, 2CS is stable for $s \in [-0.30, 4.5]$ (here the parameter s

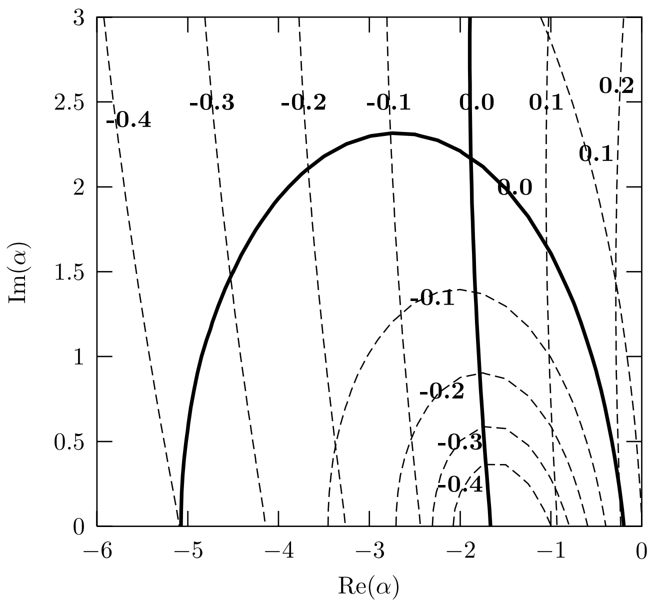

Fig. 4. The contour plots of the master stability function (the largest Lyapunov exponent) are shown in the complex α plane for coupled Rössler and Lorenz oscillators with $a = -0.5, b = 0.25, c = 2.5, d = -2.5$. There are two sets of contours corresponding to Eqs. (18). The bold solid contours are for the zero conditional Lyapunov exponent and determine the stability bounds of 2CS.

is included in the cluster for Rössler systems). We have also verified our lower bound for randomly generated coupling matrices.

5. Conclusion

In this paper, we have derived the existence conditions and stability bounds for multicluster synchronization. The stability bounds are expressed in the form of conditional master stability function so that they can be applied to a wide variety of networks and coupling strengths. We demonstrated the existence of multicluster synchronization both in identical oscillators as well as different oscillators. For identical Rössler oscillators, we find that in certain parameter range, 1CS is unstable but 2CS can be stable and this is a only a consequence of network properties. Also, our analysis can help in designing networks that can lead to multicluster synchronization.

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Appendix A

Transverse Eigenvectors

Here we consider the properties of the transverse eigenvectors of the coupling matrix G. Let us first consider the case of 1CS when G satisfies the condition (2). As noted earlier, this condition ensures that G has one eigenvector of the type $e_1^R = (1, \ldots, 1)^T$ with eigenvalue $\gamma_1 = g$ and it defines the synchronization manifold. The left eigenvectors corresponding to the transverse manifold will, in general, satisfy the orthogonality relation

$$(e_k^L)^\dagger e_1^R = 0, \quad k = 2, \ldots, N. \tag{A.1}$$

Thus, the components of e_k^L satisfy the relation

$$\sum_{j=1}^{N} e_{kj}^L = 0. \tag{A.2}$$

We now consider the 2CS conditions (9). Here, G has two linearly independent eigenvectors of the form $e_i^R = (\mu, \ldots, \mu, \nu, \ldots, \nu)^T, i = 1, 2$ and they define the synchronization manifold. Now, the left eigenvectors corresponding to the transverse manifold will satisfy the orthogonality relation

$$(e_k^L)^\dagger e_i^R = 0, \;\; i = 1, 2 \;\text{ and } \;\; k = 3, \ldots, N. \tag{A.3}$$

Now, if we write the left eigenvectors in the form $e_k^L = (e_{k1}^T, e_{k2}^T)^T$, it is possible to show that for the transverse eigenvectors

$$\begin{aligned} \sum_{j=1}^{N_1} e_{k1j} &= 0, \\ \sum_{j=1}^{N_2} e_{k2j} &= 0 \end{aligned} \tag{A.4}$$

Thus the two components e_{k1} and e_{k2} corresponding to the two clusters have a property similar to that of a single cluster [Eq. (A.2)]. Hence, by separating the weight factors p_1 and p_2, the functions ϕ and ψ in Eqs. (15) should have properties similar to that of an independent single cluster. This allows us to treat them as corresponding to the individual clusters for the separable case as in Eqs. (18).

Appendix B

Pseudo-Inverse Solution

Here, we discuss an approximate solution of Eq. (17) using Moore–Penrose pseudo-inverse [Kaipo & Somersalo, 2005]. Rewrite Eq. (17) in the form

$$\begin{aligned} M \begin{pmatrix} \dot{\phi} \\ \dot{\psi} \end{pmatrix} &= \begin{pmatrix} p_1 I & p_2 I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\phi} \\ \dot{\psi} \end{pmatrix} \\ &= \begin{pmatrix} p_1 D_\phi & p_2 D_\psi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} \end{aligned} \tag{B.1}$$

where $D_\phi = [D\mathbf{f} + \alpha D\mathbf{u}]$ and $D_\psi = [D\mathbf{g} + \alpha D\mathbf{v}]$. Using the pseodoinverse of M the above equation becomes

$$\begin{pmatrix} \dot{\phi} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} p_1^2 D_\phi & p_1 p_2 D_\psi \\ p_1 p_2 D_\phi & p_2^2 D_\psi \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix}. \tag{B.2}$$

A general solution of Eq. (17) is obtained by adding to the peudoinverse solution of the above equation, any vector $(\phi', \psi')^T$ which belongs to the null space of M i.e. $p_1 \dot{\phi}' + p_2 \dot{\psi}' = 0$. Clearly, the added vector is not important for calculating the Lyapunov exponents and hence the pseudo-inverse solution gives a reasonable approximation.

For $m = 1$, Eq. (19) can be further simplified by noting that the matrix on the RHS of Eq. (19) has the eigenvalues $p_1^2 D_\phi + p_2^2 D_\psi$ and zero and the corresponding eigenvectors are $v_1 = (1, p_2/p_1)^T$ (excluding the separable $p_1 = 0$ case) and $v_2 = (1, -(p_1/p_2)(D_\psi)^{-1}D_\phi)^T$. We note that the ratio of the two components of v_2 changes with time, but since $\dot{v}_2 = 0$, the eigenvector v_2 is not relevant for calculating the Lyapunov exponents. On the other hand, the two components of v_1 are in the ratio p_2/p_1 and the ratio does not change with time. Thus, we can average over time and obtain the Lyaponov exponent as

$$\langle p_1^2 D_\phi + p_2^2 D_\psi \rangle.$$

This allows us to express the master stability function (MSF) (or the maximum Lyapunov exponent for the entire network) as

$$\lambda_{\max} = p_1^2 \lambda_\phi + p_2^2 \lambda_\psi, \tag{B.3}$$

where λ_ϕ and λ_ψ are the CMSF for the two clusters obtained from $\langle D_\phi \rangle$ and $\langle D_\psi \rangle$, i.e. Eqs. (18).

Unfortunately, for $m > 1$, Eq. (B.3) is not valid and can only serve as a not very useful upper bound. However, Eq. (19) can still be used to get an approximate solution.