Representations of Sp(6,ℝ) and SU(3) carried by homogeneous polynomials

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In this paper, we study representations of Sp(6,ℝ) and SU(3) carried by homogeneous polynomials of phase-space variables in six dimensions. These representations are very important for the study of symplectic integration techniques for Hamiltonian systems. We obtain irreducible representations for Sp(6,ℝ) and SU(3) and explicit expressions for states within SU(3) representations in terms of phase-space variables. © *1997 American Institute of Physics.* [S0022-2488(97)05005-6]

I. INTRODUCTION

In the study of nonlinear Hamiltonian dynamics, the real symplectic group $Sp(2n,ℝ)$ and its compact subgroups play an important role.[1,2] Quite often, one studies the single particle dynamics of nonlinear Hamiltonian systems. Since this has three degrees of freedom, the relevant group is $Sp(6,ℝ)$. Moreover, the equations of motion are formulated in terms of phase-space variables (generalized coordinates and momenta). In particular, in Lie perturbation theory[1] of Hamiltonian dynamics, homogeneous polynomials of phase-space variables play a central role. Therefore, it is important to study the representations of $Sp(6,ℝ)$ carried by these polynomials. Further, in deriving symplectic integration algorithms[3–6] for Hamiltonian systems, representations of compact subgroups of $Sp(6,ℝ)$ (especially SU(3)) carried by homogeneous polynomials are required. They may also be useful in deriving metric invariants for symplectic maps.[2] For these reasons, we study the representations of $Sp(6,ℝ)$ and SU(3) carried by homogeneous polynomials of phase-space variables.

In Section II, we introduce the mathematical preliminaries. In Section III, we study the irreducible representations of $Sp(6,ℝ)$ (and its associated Lie algebra $sp(6,ℝ)$) carried by homogeneous polynomials of phase-space variables. In Section IV, we study the irreducible representations of SU(3) carried by these polynomials. We give explicit expressions for states within the representations in terms of phase-space variables in Appendix A. Such expressions are crucial in developing symplectic integration algorithms.[4] Concluding remarks can be found in Section V.

II. PRELIMINARIES

We start by defining Lie operators. Let us denote the collection of six phase-space variables q_i, p_i ($i = 1,2,3$) by the symbol z:

$$z = (q_1, p_1, q_2, p_2, q_3, p_3). \tag{2.1}$$

The Lie operator corresponding to a phase-space function $f(z)$ is denoted by: $f(z)$:. It is defined[1] by its action on a phase-space function $g(z)$ as shown below:

$$:f(z):g(z) = [f(z), g(z)], \tag{2.2}$$

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where $[f(z),g(z)]$ denotes the usual Poisson bracket of the functions $f(z)$ and $g(z)$. In particular, if the Lie operator corresponding to a homogeneous polynomial f_2 of degree 2 acts on a homogeneous polynomial g_m of degree m, it gives back another homogeneous polynomial h_m of degree m:

$$:f_2:g_m = h_m. \tag{2.3}$$

We next define the exponential of a Lie operator. It is called a Lie transformation and is given as follows:[1]

$$e^{:f(z):} = \sum_{n=0}^{\infty} \frac{:f(z):^n}{n!}. \tag{2.4}$$

Let M be a 6×6 real symplectic matrix. That is, it satisfies the following symplectic condition

$$\widetilde{M}JM = J, \tag{2.5}$$

where $\widetilde{M}$ is the transpose of M and J is the fundamental symplectic matrix.[1] The set of all such matrices forms the finite dimensional real symplectic group Sp(6,ℝ). We also have the following relation between symplectic matrices and Lie transformations:[1]

$$e^{:f_2^{(c)}:}e^{:f_2^{(a)}:}z_i = \sum_{j=1}^{6} M_{ij}z_j \equiv (Mz)_i, \tag{2.6}$$

i.e., given any symplectic matrix M, one can find two unique second degree homogeneous polynomials $f_2^{(c)}$ and $f_2^{(a)}$ such that the above relation is satisfied.

Finally, the set of all $:f_2:$'s gives a realization the Lie algebra sp(6,ℝ)[1] if we define the Lie product of two Lie operators $:f_2:$ and $:g_2:$ to be their commutator $\{:f_2:,:g_2:\}$. This commutator can be shown to satisfy the relation

$$\{:f_2:,:g_2:\} \equiv :f_2::g_2: - :g_2::f_2: = :[f_2,g_2]:. \tag{2.7}$$

III. REPRESENTATIONS OF sp(6,ℝ) AND Sp(6,ℝ)

First, we study the representations of the symplectic algebra sp(6,ℝ). These representations are obtained by the action of Lie operators on carrier spaces spanned by homogeneous polynomials. They are shown to be irreducible and correspond to the representation $(m,0,0)$ where m is the degree of the homogeneous polynomial. Next, we study the representations of the symplectic group Sp(6,ℝ) obtained by the action of linear Lie transformations on homogeneous polynomials. We end by proving a couple of relations linking the representations of sp(6,ℝ) with representations of Sp(6,ℝ).

A. Representation of sp(6,ℝ)

We have seen that the $:f_2:$'s constitute the symplectic Lie algebra sp(6,ℝ). An N dimensional representation of sp(6,ℝ) is obtained by mapping each element $:f_2:$ onto a $N \times N$ matrix $d(f_2)$ such that the following conditions are satisfied for all $:f_2:$, $:g_2:$ belonging to sp(6,ℝ):[7]

$$d(af_2 + bg_2) = ad(f_2) + bd(g_2), \quad a,b \in \mathbb{R}, \tag{3.1}$$

$$d([f_2,g_2]) = \{d(g_2),d(f_2)\}. \tag{3.2}$$

Irreducible representations of sp(6,ℝ) carried by homogeneous polynomials in z can be obtained as follows. Let $\mathscr{P}^{(m)}(z)$ denote the set of all homogeneous polynomials in z of degree m. From Eq. (2.3), we get the following relation:

$$:f_2:g_m=[f_2,g_m]\in\mathscr{P}^{(m)}\quad\forall\,g_m\in\mathscr{P}^{(m)}.\tag{3.3}$$

That is, the set of all elements belonging to sp(6,ℝ) leaves $\mathscr{P}^{(m)}$ invariant.

Let $\{P_\alpha^{(m)}\}$ be a basis for the set $\mathscr{P}^{(m)}$. Typically, we choose these basis elements to be the monomials of degree m in the six phase-space variables. The number of basis monomials $N(m)$ of degree m in the six phase-space variables is given by the relation[8]

$$N(m)=\binom{m+5}{m}.\tag{3.4}$$

From Eq. (3.3) and the completeness of the set $\{P_\alpha^{(m)}\}$, we get the following relation

$$:f_2:P_\alpha^{(m)}(z)=d^{(m)}(f_2)_\alpha^\beta P_\beta^{(m)}(z)\quad\alpha=1,2,\ldots,N(m),\tag{3.5}$$

where $d^{(m)}(f_2)_\alpha^\beta$ are coefficients multiplying the basis elements. Here we have used Einstein's summation convention. This convention will be used throughout the rest of the paper unless stated otherwise.

As we vary α from 1 to $N(m)$ in Eq. (3.5), the set of coefficients $d^{(m)}(f_2)_\alpha^\beta$ gives rise to an $N(m)\times N(m)$ matrix, $d^{(m)}(f_2)$, for each $:f_2:$ belonging to sp(6,ℝ). We claim that the set of such matrices (obtained by letting $:f_2:$ range over the entire Lie algebra) gives an $N(m)$-dimensional representation of $sp(6,ℝ)$. To prove this, we have to verify that these matrices satisfy Eqs. (3.1) and (3.2). From Eq. (2.7), we obtain the relations

$$:af_2+bg_2:P_\alpha^{(m)}(z)=a:f_2:P_\alpha^{(m)}(z)+b:g_2:P_\alpha^{(m)}(z),\quad a,b\in\mathbb{R},\tag{3.6}$$

$$:[f_2,g_2]:P_\alpha^{(m)}(z)=:f_2::g_2:P_\alpha^{(m)}(z)-:g_2::f_2:P_\alpha^{(m)}(z).\tag{3.7}$$

Substituting Eq. (3.5) into these equations, we get the desired results.

We next prove that the above representations $d^{(m)}(f_2)$ (for each m) are irreducible.

Theorem 1: The representation $d^{(m)}(f_2)$ of the Lie algebra sp(6,ℝ) is irreducible.

Proof: We note that $\mathscr{P}^{(m)}(z)$ acts as a carrier space for the Lie operators $:f_2:(\in\text{sp}(6,ℝ))$. The representation $d^{(m)}(f_2)$ is shown to be irreducible by proving that any invariant subspace $S^{(m)}$ of $\mathscr{P}^{(m)}(z)$ has to be a trivial subspace. If the given subspace $S^{(m)}$ contains only the identity element (given by 0), it is already a trivial subspace and we are done. Therefore, assume that the given invariant subspace $S^{(m)}$ has at least one element g other than the identity.

Lemma 1: q_1^m is an element of the invariant subspace $S^{(m)}$.

Proof: The element g can be decomposed in terms of the linearly independent basis elements $P_i^{(m)}$ of $\mathscr{P}^{(m)}(z)$ as follows:

$$g=A_iP_i^{(m)},\tag{3.8}$$

where

$$P_i^{(m)}=q_1^{a_i}p_1^{b_i}q_2^{c_i}p_2^{d_i}q_3^{e_i}p_3^{f_i},\quad i=1,2,\ldots,N(m)\tag{3.9}$$

and

$$a_i+b_i+c_i+d_i+e_i+f_i=m.\tag{3.10}$$

Here $N(m)$ is the dimension of the carrier space $\mathscr{P}^{(m)}(z)$ and is given by Eq. (3.4). The quantities A_i are constants.

First, we pick a unique monomial $P_*^{(m)}$ from among the basis monomials $P_i^{(m)}$ in the expansion for g as follows. Consider the following sequence of nested sets:

$$\Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \Gamma_3 \supseteq \Gamma_4 \supseteq \Gamma_5, \tag{3.11}$$

where [cf. Eq. (3.9)]

$$\Gamma_0 = \{i : A_i \neq 0\},$$

$$\Gamma_1 = \{i : b_i \geq b_j, \forall j \in \Gamma_0\},$$

$$\Gamma_2 = \{i : c_i \geq c_j, \forall j \in \Gamma_1\},$$

$$\Gamma_3 = \{i : d_i \geq d_j, \forall j \in \Gamma_2\}, \tag{3.12}$$

$$\Gamma_4 = \{i : e_i \geq e_j, \forall j \in \Gamma_3\},$$

$$\Gamma_5 = \{i : f_i \geq f_j, \forall j \in \Gamma_4\}.$$

Note that elements of the above sets Γ_j ($j = 0, 1 \ldots, 5$) are nothing but the indices that uniquely label the basis monomials.

It is easy to see that Γ_5 contains only a single element. If this were not true, Γ_5 would contain at least two distinct elements i and j. This would imply that there are basis elements $P_i^{(m)}$ and $P_j^{(m)}$ such that the following condition is satisfied:

$$b_i = b_j, \quad c_i = c_j, \quad d_i = d_j, \quad e_i = e_j, \quad f_i = f_j. \tag{3.13}$$

This in turn implies that a_i and a_j are also equal [cf. Eq. (3.10)]. Therefore, $P_i^{(m)}$ and $P_j^{(m)}$ would be equal even though their indices i and j are different. This contradicts our assumption that they are linearly independent. The above argument proves that one of the subsets Γ_5 has a single element. Of course, it is possible that one of the Γ_l's (for $l < 5$) already contains only a single element. In that case, all subsequent Γ_j's (for $j > l$) will also have the same single element. In particular, Γ_5 will have a single element which is what we require.

Let us denote the unique basis element corresponding to the only element of Γ_n by $P_*^{(m)}$, i.e.,

$$P_*^{(m)} = P_i^{(m)}(i \in \Gamma_n) = q_1^{a_*} p_1^{b_*} q_2^{c_*} p_2^{d_*} q_3^{e_*} p_3^{f_*}. \tag{3.14}$$

Further, let us denote the coefficient associated with $P_*^{(m)}$ in the decomposition of g [cf. Eq. (3.8)] by A_*. It is then easy to see that the following equation is satisfied:

$$\left[\frac{(-1)^{c_*+e_*}}{2^{b_*} A_*} \frac{:q_1 q_3 :^{f_*}}{f_*!} \frac{:q_1 p_3 :^{e_*}}{e_*!} \frac{:q_1 q_2 :^{d_*}}{d_*!} \frac{:q_1 p_2 :^{c_*}}{c_*!} \frac{:q_1^2 :^{b_*}}{b_*!} \right] g = q_1^m. \tag{3.15}$$

Since $S^{(m)}$ is assumed to be an invariant subspace under the action of sp(6,ℝ), the quantity obtained by successive actions of $:f_2:$'s on g is also an element of $S^{(m)}$. Therefore, Eq. (3.15) shows that q_1^m is an element of $S^{(m)}$. This proves the lemma.

It can now be shown that any arbitrary element h of $\mathscr{P}^{(m)}$ is also an element of $S^{(m)}$. Decompose this element as follows:

$$h = B_j P_j^{(m)}. \tag{3.16}$$

where the basis monomials $P_j^{(m)}$ are specified by Eqs. (3.9) and (3.10). Then there exists an operator that takes the element q_1^m into h as shown below:

$$\sum_{j=1}^{N}\left[B_j\,\frac{a_j!c_j!e_j!}{m!}\,\frac{(-1)^{b_j+c_j+f_j}}{2^{b_j+d_j+f_j}}\,:p_3^2:^{f_j}\,\frac{:q_3p_2:^{(e_j+f_j)}}{(e_j+f_j)!},:p_2^2:^{d_j}\,\frac{:q_2p_1:^{(m-a_j-b_j)}}{(m-a_j-b_j)!}\,:p_1^2:^{b_j}\right]q_1^m=h.$$

$$(3.17)$$

Thus we have shown that it is possible to map the element g belonging to $S^{(m)}$ into an arbitrary element of $\mathscr{P}^{(m)}(z)$ by the action of an appropriate combination of elements belonging to sp(6,ℝ). Since $S^{(m)}$ was assumed to be an invariant subspace under the action of sp(6,ℝ), this proves that every element of $\mathscr{P}^{(m)}(z)$ is also an element of $S^{(m)}$. Therefore, if $S^{(m)}$ is an invariant subspace, it has to be a trivial subspace of $\mathscr{P}^{(m)}$. Hence $d^{(m)}(f_2)$ is an irreducible representation of sp(6,ℝ).

B. Representation of Sp(6,ℝ)

Irreducible representations of Sp(6,ℝ) carried by homogeneous polynomials in z can be obtained by once again utilizing the set $\mathscr{P}^{(m)}(z)$. Consider the action of Sp(6,ℝ) on this set. Using Eq. (3.3), the following relation is seen to be true:

$$e^{:f_2^{(c)}:}e^{:f_2^{(a)}:}g_m\in\mathscr{P}^{(m)}\quad\forall\,g_m\in\mathscr{P}^{(m)}.\qquad(3.18)$$

That is, the set of all symplectic matrices leaves $\mathscr{P}^{(m)}$ invariant [cf. Eq. (2.6)]. We again choose $\{P_\alpha^{(m)}\}$ to be the set of basis elements for $\mathscr{P}^{(m)}(z)$. From Eq. (2.6) we get the following result:

$$e^{:f_2^{(c)}:}e^{:f_2^{(a)}:}P_\alpha^{(m)}(z)=P_\alpha^{(m)}(Mz)=\mathscr{D}^{(m)}(M)_\alpha^\beta P_\beta^{(m)}(z)\quad\alpha=1,2,\cdots,N(m),\qquad(3.19)$$

where $\mathscr{D}^{(m)}(M)_\alpha^\beta$ is the coefficient corresponding to $P_\beta^{(m)}(z)$.

As α is varied from 1 to $N(m)$, the set of coefficients $\mathscr{D}^{(m)}(M)_\alpha^\beta$ gives rise to an $N(m)\times N(m)$ matrix $\mathscr{D}^{(m)}(M)$ for each M belonging to Sp(6,ℝ). We claim that this set of matrices gives an $N(m)$-dimensional representation of Sp(6,ℝ). Consider the following quantity:

$$(e^{:g_2^{(c)}:}e^{:g_2^{(a)}:})(e^{:f_2^{(c)}:}e^{:f_2^{(a)}:})P_\alpha^{(m)}(z).\qquad(3.20)$$

Denote the symplectic matrices corresponding to the two factors in the above equation by M and M' [cf. Eq. (2.6)]. From Eq. (3.19), we get the following relation:

$$(e^{:g_2^{(c)}:}e^{:g_2^{(a)}:})(e^{:f_2^{(c)}:}e^{:f_2^{(a)}:})P_\alpha^{(m)}(z)=\mathscr{D}^{(m)}(MM')_\alpha^\gamma P_\gamma^{(m)}(z).\qquad(3.21)$$

We can also evaluate the actions of the two factors on the basis element one after the other to obtain

$$(e^{:g_2^{(c)}:}e^{:g_2^{(a)}:})(e^{:f_2^{(c)}:}e^{:f_2^{(a)}:})P_\alpha^{(m)}(z)=\mathscr{D}^{(m)}(M)_\alpha^\beta\mathscr{D}^{(m)}(M')_\beta^\gamma P_\gamma^{(m)}(z).\qquad(3.22)$$

Comparing Eqs. (3.21) and (3.22), we obtain the desired relation

$$\mathscr{D}^{(m)}(MM')=\mathscr{D}^{(m)}(M)\mathscr{D}^{(m)}(M').\qquad(3.23)$$

Next, we show that these representations are irreducible. For sp(6,ℝ), we proved that there are no non-trivial subspaces of $\mathscr{P}^{(m)}$ that are invariant under the action of the $:f_2:$'s. From Eqs. (2.6) and (2.4), it follows that there are no non-trivial invariant subspaces of $\mathscr{P}^{(m)}$ even under the action of symplectic matrices M belonging to Sp(6,ℝ). Therefore, the representations $\mathscr{D}^{(m)}(M)$ are irreducible. In fact, they correspond to the irreducible representation $(m,0,0)$ of Sp(6,ℝ)^2.

We end this subsection by giving examples of the irreducible representations of Sp(6,ℝ). It is obvious that the M's themselves form a six dimensional irreducible representation. This is called the fundamental (or defining) representation. Formally, it can be obtained from Eq. (3.19) by setting m equal to 1. Another important irreducible representation is the adjoint representation $\mathscr{D}^{(2)}(M)$ obtained by setting m equal to 2 in Eq. (3.19). From Eq. (3.4), it is seen that this forms a 21 dimensional irreducible representation of Sp(6,ℝ).

In the above discussions, we have been careful to distinguish between upper and lower indices labeling the matrix elements of the representation. This is because an $N(m)$-dimensional representation is (in general) not equivalent to its own transpose, i.e., $\widetilde{D}^{(m)}(M)$ is not a member of the representation $\mathscr{D}^{(m)}$. For the fundamental representation ($m = 1$), it turns out that this distinction is unnecessary since $\widetilde{M}$ also belongs to the group of symplectic matrices.

C. Relations between representations of sp(6,ℝ) and Sp(6,ℝ)

One relation between the representation $\mathscr{D}^{(m)}(M)$ of the group and the representation $d^{(m)}(f_2)$ of the algebra is given as follows (provided M sufficiently close to the identity, in which case, the two Lie transformations appearing in Eq. (2.6) can be combined into one):

$$\mathscr{D}^{(m)}(M) = \mathscr{D}^{(m)}(e^{:f_2:}) = e^{d^{(m)}(f_2)}. \tag{3.24}$$

Another interesting relation between the representations of Sp(6,ℝ) and sp(6,ℝ) is given by the following theorem.

Theorem 2: *Denote the set of basis elements for $\mathscr{P}^{(2)}$ by $\{w_\alpha\}$. Then*

$$\mathscr{D}^{(2)}(M)_\alpha^\beta d^{(m)}(w_\beta) = \mathscr{D}^{(m)}(M)^{-1} d^{(m)}(w_\alpha) \mathscr{D}^{(m)}(M). \tag{3.25}$$

Proof: It is clear that the set $\{w_\alpha\}$ is identical to the set $\{P_\alpha^{(2)}(z)\}$. In fact, the new notation was adopted merely for notational convenience. From Eq. (3.19), we get the following result:

$$:e^{:f_2^{(c)}:}e^{:f_2^{(a)}:}w_\alpha: = \mathscr{D}^{(2)}(M)_\alpha^\beta:w_\beta:. \tag{3.26}$$

We also obtain the relation

$$:e^{:f_2^{(c)}:}e^{:f_2^{(a)}:}w_\alpha: = e^{:f_2^{(c)}:}e^{:f_2^{(a)}:}:w_\alpha:e^{-:f_2^{(a)}:}e^{-:f_2^{(c)}:}. \tag{3.27}$$

Comparing the last two equations, we find the result

$$\mathscr{D}^{(2)}(M)_\alpha^\beta:w_\beta: = e^{:f_2^{(c)}:}e^{:f_2^{(a)}:}:w_\alpha:e^{-:f_2^{(a)}:}e^{-:f_2^{(c)}:}. \tag{3.28}$$

When the left and right hand sides of the above equation act on the basis element $P_\gamma^{(m)}(z)$, we get the following relations [cf. Eqs. (3.5) and (3.19)]:

$$\mathscr{D}^{(2)}(M)_\alpha^\beta:w_\beta:P_\gamma^{(m)}(z) = \mathscr{D}^{(2)}(M)_\alpha^\beta d^{(m)}(w_\beta)_\gamma^\nu P_\nu^{(m)}(z), \tag{3.29}$$

$$e^{:f_2^{(c)}:}e^{:f_2^{(a)}:}:w_\alpha:e^{-:f_2^{(a)}:}e^{-:f_2^{(c)}:}P_\gamma^{(m)}(z) = [\mathscr{D}^{(m)}(M^{-1})d^{(m)}(w_\alpha)\mathscr{D}^{(m)}(M)]_\gamma^\nu P_\nu^{(m)}(z). \tag{3.30}$$

We also have the following standard result:

$$\mathscr{D}^{(m)}(M^{-1}) = [\mathscr{D}^{(m)}(M)]^{-1}. \tag{3.31}$$

Inserting Eqs. (3.29), (3.30), and (3.31) into Eq. (3.28), we get the desired result.

IV. REPRESENTATIONS OF SU(3)

This section is devoted to study of representations of SU(3) carried by homogeneous polynomials in the phase-space variables. First, we briefly review relevant aspects of the representation theory of SU(3). Next, we list irreducible representations of SU(3) carried by mth degree homogeneous polynomials. Finally, we list the weight vectors within each irreducible representation carried by homogeneous polynomials of degree less than five.

Irreducible representations of SU(3) are labeled by two indices j_1 and j_2.[7] The dimension of the irreducible representation labeled by (j_1,j_2) is given as follows:

$$N(j_1,j_2) = \tfrac{1}{2}(j_1+1)(j_2+1)(j_1+j_2+2). \tag{4.1}$$

States within an irreducible representation are labeled by I (total isotopic spin), I_3 (the third component of isotopic spin) and Y with hypercharge. We will denote the states within this representation as follows:

$$|j_1,j_2;I,I_3,Y\rangle. \tag{4.2}$$

Here, we have abused notation to denote the eigenvalues corresponding to a operator by the symbol used to denote the operator itself.

We now turn to the problem of determining the representations of SU(3) carried by homogeneous polynomials in the phase space variables. In the previous section, we have already seen that homogeneous polynomials of degree m carry the irreducible representation $(m,0,0)$ of Sp(6,$\mathbb{R}$). Under the action of SU(3), this representation will, in general, be reducible. But, it can be written as a direct sum of irreducible representations of SU(3). This list of irreducible representations of SU(3) constitutes the ''branching rule'' of $(m,0,0)$. The required branching rule is given as follows:[9]

Theorem 3: *The complete list of irreducible representations of* SU(3) *carried by homogeneous polynomials of degree* m *in phase space variables is given as follows*:

$$(m,0),(m-1,1),\ldots,(1,m-1),(0,m),$$

$$(m-2,0),(m-3,1),\ldots,(1,m-3),(0,m-3),$$

$$\ldots$$

$$\ldots$$

$$(0,0) \quad \text{if } m \text{ is even}$$

$$\text{or } (1,0),(0,1) \quad \text{if } m \text{ is odd.} \tag{4.3}$$

We next turn our attention to the weight vectors within each such representation (also called states of a representation or basis vectors of SU(3)). It can be shown[10–12] that these states are associated with harmonic functions on the 5-sphere S^5. The 5-sphere is defined by the relation

$$z_1^* z_1 + z_2^* z_2 + z_3^* z_3 = r^2 = 1, \tag{4.4}$$

where z_j and z_j^* are given by the relations

$$z_j = \frac{1}{\sqrt{2}}(q_j + ip_j), \tag{4.5}$$

$$z_j^* = \frac{1}{\sqrt{2}} (q_j - ip_j). \tag{4.6}$$

Since we are interested in functions defined on the 5-sphere S^5, it is convenient to parametrize S^5 in terms of polar coordinates ϕ_1, ϕ_2, ϕ_3, θ and ξ. These coordinates are related to the complex phase-space variables z_j by the following relations

$$z_1 = re^{i\phi_1} \cos \theta, \tag{4.7}$$

$$z_2 = re^{i\phi_2} \sin \theta \cos \xi, \tag{4.8}$$

$$z_3 = re^{i\phi_3} \sin \theta \sin \xi, \tag{4.9}$$

where

$$0 \leq \phi_1, \phi_2, \phi_3 \leq 2\pi; \quad 0 \leq \theta, \xi \leq \pi/2. \tag{4.10}$$

It can be shown[10] that states within the irreducible representation (j_1, j_2) of SU(3) can be associated with harmonic functions defined on S^5 as shown below:

$$|j_1, j_2; I, I_3, Y\rangle = \frac{1}{\sin \theta} d^{1/2(j_1+j_2+1)}_{1/6(j_1-j_2-3Y+6I+3), 1/6(j_1-j_2-3Y-6I-3)}(2\theta) d^{(I)}_{1/3(j_1-j_2)+1/2Y, I_3}$$

$$\times (2\xi) e^{1/3i(j_1-j_2)(\phi_1+\phi_2+\phi_3)} e^{iI_3(\phi_2-\phi_3)} e^{1/2iY(-2\phi_1+\phi_2+\phi_3)}. \tag{4.11}$$

Here $d^{(j)}_{m',m}(\beta)$ are the usual d-functions that characterize the irreducible representation (j) of SU(2). The sign convention for the d-function is taken to be that given in Edmonds,[13] i.e.,

$$d^{(j)}_{m',m}(\beta) = \langle jm'|\exp(+i\beta J_y/h)|jm\rangle, \tag{4.12}$$

where $|jm\rangle$ denotes states within the representation (j) of SU(2).

We are now in a position to give explicit formulas for the states within the representations of SU(3) carried by f_n. Such expressions are necessary to construct some of the symplectic integration algorithms.[4–6] These formulas are listed in Appendix A (due to lack of space, only expressions for small values of n are given). These are obtained using the basis functions introduced earlier [cf. Eq. (4.11)]. However, we multiply these basis functions (which are dimensionless) by r^n in order to get the dimensions properly.[14] Thus we use the basis functions:

$$|n; j_1, j_2; I, I_3, Y\rangle \equiv r^n |j_1, j_2; I, I_3, Y\rangle. \tag{4.13}$$

This multiplication does not change the eigenvalues I, I_3 or Y. Moreover, the states within the representation are given in terms of z_1, z_2, z_3 [cf. Eqs. (4.7), (4.8), and (4.9)] and their complex conjugates instead of the original angular variables. This makes identification with the homogeneous polynomials easier. Each entry in Appendix A take the following general form:

$$I, I_3, Y, |n; j_1, j_2; I, I_3, Y\rangle (z_1, z_1^*, z_2, z_2^*, z_3, z_3^*). \tag{4.14}$$

We only list the states within the representations for which j_1 is greater than or equal to j_2. Given a state $|n; j_1, j_2; I, I_3, Y\rangle$ belonging to (j_1, j_2), the corresponding state belonging to (j_2, j_1) is given by $|n; j_2, j_1; I, I_3, -Y\rangle$. Moreover, it satisfies the following relation:

$$|n; j_2, j_1; I, I_3, -Y\rangle (z_1, z_1^*, z_2, z_2^*, z_3, z_3^*) = (-1)^{I+I_3} |n; j_1, j_2; I, I_3, Y\rangle (z_1^*, z_1, z_3^*, z_3, z_2^*, z_2). \tag{4.15}$$

Therefore, given the states within the representation (j_1, j_2), the states within (j_2, j_1) are easily obtained.

V. SUMMARY

In this paper, we looked at representations of Sp(6,ℝ) and SU(3) carried by homogeneous polynomials in six phase-space variables. It was shown that homogeneous polynomials of degree m carry a $N(m)$ [cf. Eq. (3.4)] dimensional irreducible representation of Sp(6,ℝ). These irreducible representations break into a direct sum of irreducible representations for SU(3). Explicit expressions for SU(3) states within these representations were given in terms of phase variables. The above results should be useful in Lie perturbation theory of symplectic maps, especially in the theory of symplectic integration.

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APPENDIX A: REPRESENTATIONS OF SU(3) CARRIED BY HOMOGENEOUS POLYNOMIALS

A. Representations of SU(3) carried by f_0

I. $j_1 = 0$, $j_2 = 0$ (one-dimensional irreducible representation)

I	I_3	Y	$|n;j_1,j_2;I,I_3,Y\rangle$
0	0	0	r^0

B. Representations of SU(3) carried by f_1

I. $j_1 = 1$, $j_2 = 0$ (three-dimensional irreducible representation)

I	I_3	Y	$|n;j_1,j_2;I,I_3,Y\rangle$
1/2	1/2	1/3	z_2
1/2	$-1/2$	1/3	z_3
0	0	$-2/3$	$\sqrt{2}z_1$

C. Representations of SU(3) carried by f_2

I. $j_1 = 2$, $j_2 = 0$ (six-dimensional irreducible representation)

I	I_3	Y	$|n;j_1,j_2;I,I_3,Y\rangle$
1	1	2/3	z_2^2
1	0	2/3	$\sqrt{2}z_2 z_3$
1	-1	2/3	z_3^2
1/2	1/2	$-1/3$	$\sqrt{3}z_1 z_2$
1/2	$-1/2$	$-1/3$	$\sqrt{3}z_1 z_3$
0	0	$-4/3$	$\sqrt{3}z_1^2$

II. $j_1 = 1$, $j_2 = 1$ (eight-dimensional irreducible representation)

I	I_3	Y	$\lvert n;j_1,j_2;I,I_3,Y\rangle$
1/2	1/2	1	$\sqrt{3}\,z_z^* z_2$
1/2	$-1/2$	1	$\sqrt{3}\,z_1^* z_3$
1	1	0	$-\sqrt{2}\,z_2 z_3^*$
1	0	0	$z_2^* z_2 - z_3^* z_3$
1	-1	0	$\sqrt{2}\,z_2^* z_3$
0	0	0	$2z_1^* z_1 - z_2^* z_2 - z_3^* z_3$
1/2	1/2	-1	$-\sqrt{3}\,z_1 z_3^*$
1/2	$-1/2$	-1	$\sqrt{3}\,z_1 z_2^*$

III. $j_1 = 0$, $j_2 = 0$ (one-dimensional irreducible representation)

I	I_3	Y	$\lvert n;j_1,j_2;I,I_3,Y\rangle$
0	0	0	$z_1^* z_1 + z_2^* z_2 + z_3^* z_3 = r^2$

[1] A. J. Dragt, in *Physics of High Energy Particle Accelerators*, edited by R. A. Carrigan, F. R. Huson, and M. Month, AIP Conf. Proc. No. 87 (AIP, New York, 1982), p. 147; A. J. Dragt, F. Neri, G. Rangarajan, D. R. Douglas, L. M. Healy, and R. D. Ryne, Annu. Rev. Nucl. Part. Sci. **38**, 455 (1988).
[2] G. Rangarajan, Ph.D. thesis, University of Maryland, 1990.
[3] J. Irwin, SSC Report, No. 228, 1989.
[4] G. Rangarajan, J. Math. Phys. **37**, 4514 (1996).
[5] A. J. Dragt and D. T. Abell, Int. J. Mod. Phys. A (Proc. Suppl.) **2B**, 1019 (1993).
[6] D. T. Abell, Ph.D. thesis, University of Maryland, 1995.
[7] J. Q. Chen, *Group Representation Theory for Physicists* (World Scientific, Singapore, 1989).
[8] A. Nijenhuis and H. S. Wilf, *Combinatorial Algorithms for Computers and Calculators* (Academic, New York, 1978).
[9] W. G. McKay and J. Patera, *Tables of Dimensions, Indices, and Branching Rules for Representations of Simple Lie Groups* (Marcel Dekker, New York, 1981).
[10] M. A. B. Beg and H. Ruegg, J. Math. Phys. **6**, 677 (1965).
[11] A. J. Dragt, J. Math. Phys. **6**, 533 (1965).
[12] A. J. Dragt, J. Math. Phys. **6**, 1621 (1965).
[13] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, 1957).
[14] D. T. Abell (private communication, 1995).