Transverse resistive wall effects on the dynamics of a bunched electron beam

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(Received 21 October 1988)

In the wigglers of future free-electron lasers, the electron beam will be required to travel over a length of 10 m or more in pipes with small diameters. Transverse resistive wall effects could lead to beam breakup during this transport. To investigate this possibility, the equation of motion for a bunched beam is solved analytically. Results show that a steady-state solution is reached for times larger than the diffusion time. This solution can either oscillate or grow exponentially with the length of the pipe, depending on the relative magnitudes of the resistive wall effect and the focusing force in the wiggler. The magnitude of the resistive wall effect depends on the pipe radius b (it increases as $1/b^2$) but is independent of the thickness and conductivity of the pipe. The thickness and conductivity affect only the time required to reach the steady-state solution. The possibility of a significant transient is also discussed.

I. INTRODUCTION

When a charge travels in a smooth pipe of small radius, it will generate a wake field if the pipe is not perfectly conducting. Even though the transverse force is zero immediately behind the bunch because of the cancellation of the electric force of the image charge and the magnetic force of the induced current, a finite conductivity allows the induced magnetic field to penetrate into the metal pipe with the result that the magnetic force decays more slowly than the electric force. This gives rise to a net wake-field force on the later bunches. This force increases rapidly as the radius of the pipe decreases. In future free-electron lasers (FEL's), because of efficiency requirements and limitations on achievable wiggler field, an electron beam will be required to travel in a pipe only a few millimeters in diameter over a length of 10 m or more in a wiggler. Therefore the question arises as to whether transverse resistive wall effects of the electron beam could compromise the performance of a wiggler.

Estimates of transverse resistive wall instability were done previously with formulas derived by Caporaso *et al.*[1] These formulas were derived for a dc beam and with an induced magnetic field decreasing as the square root of time, a dependence valid only for a limited time. For an FEL injected with an rf linac which has a bunched beam of long duration, these results are not appropriate. Neil and Whittum[2] have recently analyzed the case of a bunched beam. They investigated the problem in the frequency domain and used the dominant mode in the expression for the wake field. Their analysis is limited to the case where the first bunch is displaced off-axis and the subsequent bunches follow on-axis.

In this paper, an analysis of transverse resistive wall instability of a bunched electron beam in a wiggler is carried out. The full expression for the wake field is used and a complete solution is obtained analytically. Various approximations are investigated and the steady-state solution is discussed with a brief discussion of the transient state. Throughout the paper cgs units will be used.

II. TRANSVERSE RESISTIVE WALL WAKE FIELD OF A BUNCHED BEAM MOVING IN A CIRCULAR PIPE

The transverse wake field induced by a beam of relativistic particles off-axis has been investigated by Bodner *et al.*[3] The results are summarized and extended in this section.

When a dc beam current I established at time $t=0$ is traveling (in the z direction) at a distance ξ off-axis in a pipe (cf. Fig. 1) of inner radius b, outer radius d, thickness $\tau(=d-b)$, and conductivity σ, the wake field is given as a magnetic field $B_y^{\rm dc}$:

$$B_y^{\rm dc}(t)=\frac{8I\xi}{cb^2}\sum_{i=1}^{\infty}\frac{\exp(-t/T_i)}{C_i}\,, \qquad (2.1)$$

where

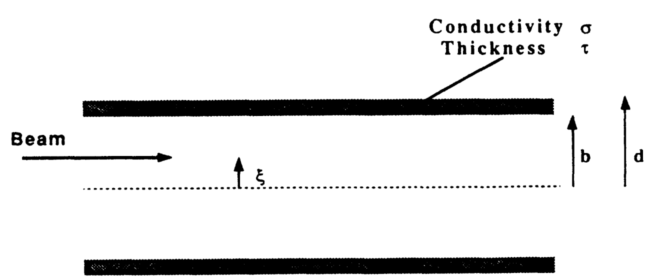

FIG. 1. Schematic diagram of the geometry used in resistive wall effect analysis.

$$T_i = \frac{4\pi\sigma b^2}{c^2 y_i^2} , \tag{2.2}$$

$$C_i = y_i^2 \left[\left[\frac{J_2(y_i)}{J_0(y_i d/b)} \right]^2 - 1 \right] , \tag{2.3}$$

and y_i's are zeros of the function

$$J_0(yd/b)N_2(y) - J_2(y)N_0(yd/b) .$$

Here J_i's and N_i's are Bessel functions of the first and second kind, respectively. The summation over i in Eq. (2.1) represents a summation over an infinite number of modes.

The expression for the magnetic field at position z and time t after a (δ function) bunch with charge q has passed, can be found from Eq. (2.1) to be[3]

$$B_y(t) = \int_{-\infty}^{t} \frac{\partial I(t')}{\partial t'} B_y^{dc}(t-t')dt' .$$

Since the bunch passes position z at $t=0$, the current it produces takes on the form $I(t) = q\delta(t)$. Carrying out the integration with this expression for I,

$$B_y(t) = -\frac{8q\xi}{cb^2} \sum_{i=1}^{\infty} \frac{\exp(-t/T_i)}{C_i T_i} . \tag{2.4}$$

The magnetic field of a bunched beam with a fixed time interval Δ between any two bunches will be a sum of the fields of the individual bunches.

In Eq. (2.4), the longest decay time is T_1. For a *thin-wall* approximation (small τ/b), T_1 is given by

$$T_1 \approx \frac{2\pi\sigma b \tau}{c^2} . \tag{2.5}$$

T_1 is usually referred to as the *diffusion time*. Table I shows the parameters used for a proposed extreme ultraviolet (XUV) FEL,[4] where the *pulse length* refers to the duration of the bunched beam. For these parameters, the *diffusion time* has a value of 0.5 μs and the corresponding first mode is the dominant mode.

An approximate formula for the magnetic field behind a (δ-function) bunch, valid for times short compared to the diffusion time, can be derived from Eq. (2.4) as

$$B_y(t) = -\frac{2q\xi}{\pi b^3(\sigma t)^{1/2}} . \tag{2.6}$$

This approximate formula was used in Ref. 1. In Fig. 2, it is compared to the exact result given in Eq. (2.4).

TABLE I. Proposed parameters for XUV FEL.

Inner pipe radius b	0.18 cm
Outer pipe radius d	0.198 cm
Conductivity of pipe σ (for titanium)	2×10^{16} s^{-1}
Wiggler field	0.75 T
Pulse length	300 ms
Electron energy	500 MeV
Length of pipe z	800 cm
Bunch separation Δ	6.8 ns
Average current I	300 mA

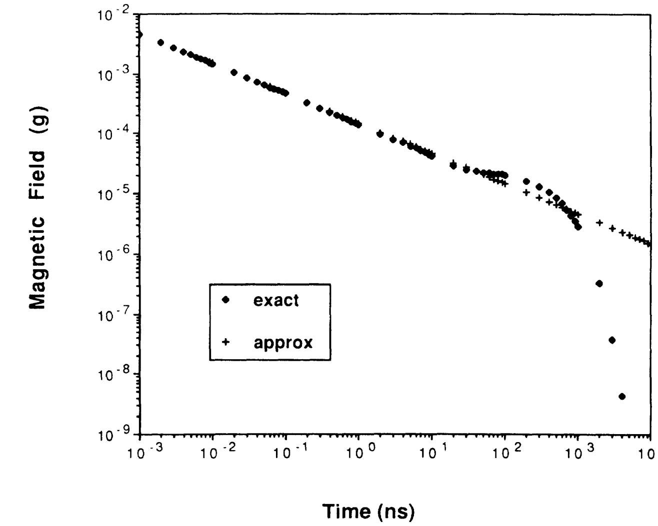

FIG. 2. Variation of the transverse resistive wall wake field B with time t elapsed since the passage of a bunch. The results obtained using the exact formula [Eq. (2.4)] are compared with those obtained using the approximate formula [Eq. (2.6)].

Equation (2.6) is a good approximation to the exact result up to one diffusion time. After one diffusion time, the magnetic field drops off rapidly as an exponential function of time because it has diffused through the pipe wall. One diffusion time is much shorter than the value of a pulse length of 300 μs (see Table I) typical of an rf-based FEL. Therefore the exact formula should be used for the analysis of most rf-based FEL's.

III. TRANSVERSE RESISTIVE WALL INSTABILITY OF A BUNCHED BEAM

The beam is considered to be a series of bunches traveling with speed c. The transverse displacement from the axis of the Kth bunch is denoted by $\xi(z=ct,K)$. Employing the Lorentz force equation and Eq. (2.4), the equation of motion for $\xi(t,K)$ is

$$\frac{d^2\xi(t,K)}{dt^2} + \omega_0^2\xi(t,K) = \sum_{i=1}^{\infty} G_i \sum_{l=0}^{K-1} \exp[-(K-l)\Delta/T_i]\xi(t,l) , \tag{3.1}$$

where

$$G_i = \frac{8eqv}{m\gamma c^2 b^2 C_i T_i} . \tag{3.2}$$

Here ω_0 is the frequency of the slow betatron motion in the wiggler, γ is the usual relativistic factor, and v is the z component of beam velocity. The sum over l represents a sum of the interactions between the Kth bunch and the wake fields of all bunches ahead of it. In Secs. III A–III F, the solution of Eq. (3.1) will be presented and various limiting cases of this solution will be investigated.

A. Complete solution

Assuming the initial transverse velocity to be zero for all bunches, Eq. (3.1) can be solved analytically using Laplace transforms as described in Appendixes A and B. The solution for $\xi(t,K)$ is found to be [cf. Eq. (A18)]

$$\tilde{\xi}(t,K)=\sum_{m=1}^{K}\xi(0,K-m)\sum_{n=1}^{m}F(n)\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_n=1}^{\infty}G_{i_1}G_{i_2}\cdots G_{i_n}\sum_{m_1=1}^{m-1}\sum_{m_2=1}^{m-m_1-1}\cdots\sum_{m_{n-1}=1}^{m-m_1-\cdots-m_{n-2}-1}$$
$$\times\exp(-m_1\Delta/T_{i_1})\exp(-m_2\Delta/T_{i_2})\cdots\exp(-m_{n-1}\Delta/T_{i_{n-1}})$$
$$\times\exp[-(m-m_1-\cdots-m_{n-1})\Delta/T_{i_n}]\ , \tag{3.3}$$

where

$$\tilde{\xi}(t,K)=\xi(t,K)-\xi(0,K)\cos(\omega_0 t) \tag{3.4}$$

and

$$F(n)=\frac{\pi^{1/2}}{2^{n+1/2}\omega_0^{2n}n!}(\omega_0 t)^{n+1/2}J_{n-1/2}(\omega_0 t)\ . \tag{3.5}$$

The above solution consists of a transient state followed by a steady state for large K. Numerical results show that the transient state is not significant for the FEL considered here[4] (cf. Sec. III E) and for the rf-based FEL considered in Ref. 2. Therefore a detailed discussion is given below only for the steady-state solution. A brief description of the transient state is given in Sec. III F.

B. Steady-state solution

If all the bunches are assumed to start out with the same transverse displacement d, it is shown in Appendix C that $\xi(t,K)$ approaches a steady state as K tends to infinity. An expression for the state-steady value $\xi(t,\infty)$ is derived. On the other hand, knowing that a steady-state solution exists allows us to present a simpler derivation of $\xi(t,\infty)$ in this section. In the limit $K\to\infty$, all $\xi(t,K)$ in Eq. (3.1) can be replaced by $\xi(t,\infty)$ to give

$$\frac{d^2\xi(t,\infty)}{dt^2}+(\omega_0^2-\Omega^2)\xi(t,\infty)=0\ , \tag{3.6}$$

where

$$\Omega^2=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}G_i\exp(-j\Delta/T_i)\ . \tag{3.7}$$

Ω^2 represents the defocusing effects of the resistive wall instability. For $\Omega^2<\omega_0^2$, Eq. (3.6) is just a betatron oscillation equation with reduced focusing. The motion of a bunch is therefore given by

$$\xi(t,\infty)=d\cos[(\omega_0^2-\Omega^2)^{1/2}t]\ . \tag{3.8}$$

The betatron frequency has changed from ω_0 to $\omega_0(1-\Omega^2/\omega_0^2)^{1/2}$. For $\Omega^2>\omega_0^2$, the defocusing effects become so large that the transverse displacement of a bunch grows exponentially with time. For the parameters given in Table I, ω_0 and Ω have values of $9.53\times10^7\ \mathrm{s}^{-1}$ and $3.06\times10^7\ \mathrm{s}^{-1}$, respectively.

Ω^2 can be proved to be independent of thickness and conductivity of the pipe. To show this, the sum over index j in Eq. (3.7) is represented by an integral over $\tilde{t}$ (where $\tilde{t}=j\Delta$). This approximation is a good one in our case since $\Delta \ll$ *pulse length* (cf. Table I). Performing the integration over $\tilde{t}$ we get

$$\Omega^2=\frac{8eIv}{m\gamma c^2b^2}\sum_{i=1}^{\infty}\frac{1}{C_i}\ .$$

Comparison of Eqs. (2.7) and (4.10) in Ref. 3 for $t=0$ gives

$$\sum_{i=1}^{\infty}\frac{1}{C_i}=\frac{1}{4}\ . \tag{3.9}$$

Therefore,

$$\Omega^2=\frac{2eIv}{m\gamma c^2b^2}\ . \tag{3.10}$$

As claimed, Ω^2 is independent of the thickness and conductivity of the pipe. It is seen to be inversely proportional to b^2.

On the other hand, the *time* required to achieve a steady-state solution *does* depend on τ and σ of the pipe. We saw earlier that the magnetic field diffuses out through the wall on a time scale of one *diffusion time* (T_1). Therefore, one would expect $\xi(t,K)$ to attain $\xi(t,\infty)$ within a few T_1's. Using $5T_1$ for specificity, we obtain the number of bunches K_∞ required to reach a steady-state solution to be

$$K_\infty\approx\frac{10\pi\sigma b\tau}{\Delta c^2}\ . \tag{3.11}$$

Thus K_∞ is directly proportional to σ, τ, and b of the pipe. For the parameters given in Table I, we estimate the value of K_∞ to be approximately 330.

C. Second-order approximation

When the resistive wall effect is small, the full solution in Eq. (3.3) can be expanded in terms of a dimensionless parameter

$$\delta_i=\frac{G_i t\exp(-\Delta/T_i)}{2\omega_0}\ . \tag{3.12}$$

If δ_i is small for all i (which is true in many practical situations), the expansion can be truncated at second order (in δ_i). For the parameter values given in Table I, $\delta_1=0.002$. With $\xi(0,j)$ equal to d for all j, the second-order solution $\xi^{(2)}(t,K)$ can be expressed as follows:

$$\xi^{(2)}(t,K)=d\cos(\omega_0 t)+d\sin(\omega_0 t)\sum_{j=1}^{K-1}\sum_{i=1}^{\infty}\delta_i\exp(-j\Delta/T_i)$$
$$+\frac{d}{2(\omega_0 t)}[\sin(\omega_0 t)-(\omega_0 t)\cos(\omega_0 t)]\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\delta_i\delta_j\sum_{m=0}^{K-2}\sum_{m_1=0}^{m}\exp[-(m-m_1)\Delta/T_i]\exp(-m_1\Delta/T_j)\ . \tag{3.13}$$

When the beam pipe thickness is small compared to the radius b, the summations over i and j usually converge rapidly and only a few terms are required. The second-order solution also has the property [as is easily seen from Eq. (3.13)] that $\xi^{(2)}(t,K)-\xi^{(2)}(t,K-1)\to 0$ as $K\to\infty$, i.e., $\xi^{(2)}(t,K)$ approaches a steady state when $K\to\infty$.

D. Single-mode approximation

There are situations when only the first mode dominates. For example, this happens when the thickness of the pipe τ is very small compared to the radius b. In such cases, a simpler expression for $\xi(t,K)$ can be obtained. The reduction from the full solution to a solution valid for one mode is described in Appendix D. The final result [cf. Eq. (D9)] is

$$\xi_1(t,K)=\sum_{k=0}^{K}\xi_1(0,K-k)\exp(-k\Delta/T_1)\sum_{n=0}^{k}\frac{G_1^nC_{k-n}^n\pi^{1/2}}{2^{n+(1/2)}\omega_0^{2n}n!}(\omega_0 t)^{n+(1/2)}J_{n-(1/2)}(\omega_0 t)\ , \tag{3.14}$$

where

$$C_l^n=(-1)^l\frac{(-n)(-n-1)\cdots(-n-l+1)}{l!}\ . \tag{3.15}$$

The steady-state value $\xi_i(t,\infty)$ is given by [cf. Eq. (D11)]

$$\xi_1(t,\infty)=d\cos[(\omega_0^2-\Omega_1^2)^{1/2}t]\ , \tag{3.16}$$

where

$$\Omega_1^2=G_1\sum_{j=1}^{\infty}\exp(-j\Delta/T_1) \tag{3.17}$$

represents the defocusing effect of the first mode. This steady-state solution can be obtained directly from Eqs. (3.7) and (3.8) by setting

$$\Omega^2=\sum_{i=1}^{\infty}\Omega_i^2\cong\Omega_1^2\ . \tag{3.18}$$

Following the line of reasoning outlined in Sec. III B, the sum over j in Eq. (3.17) can be approximated by an integral to give

$$\Omega_1^2\cong\frac{8eIv}{m\gamma c^2b^2}\frac{1}{C_1}\ , \tag{3.19}$$

where C_1 is given by Eq. (2.3). For small τ/b, an approximate value for y_1 is given by $(2b/\tau)^{1/2}$.

E. Numerical results

Equation (3.1) was also solved numerically using a computer program. $\xi(0,j)$ was assumed to be equal to 1.0 for all j. All modes were included. The results for $I=30$ and 300 mA are shown in Fig. 3. They show the transverse displacement of the beam bunches at the end of the pipe ($ct=800$ cm) as a function of time or, equivalently, bunch number. They confirm the earlier observation that $\xi(t,K)$ approaches a steady state after the passage of several bunches. The true displacement does not have the unbounded growth seen using the approximate formula for magnetic field [Eq. (2.6)] because the true magnetic field decays exponentially for times longer than one diffusion time. The adverse effects of resistive wall instability on the operation of the FEL are therefore limited. The large oscillations of bunch displacement expected to represent a transient state preceding the steady state are not observed.

Computer programs were also written to implement

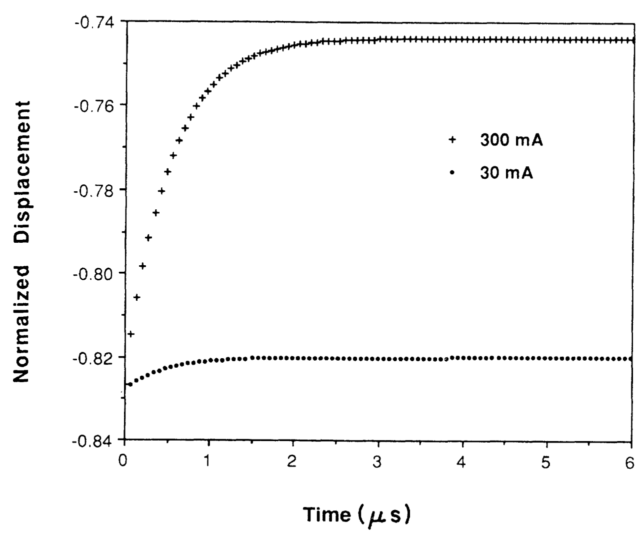

FIG. 3. The transverse displacement of the beam at the end of the wiggler as obtained by numerically solving Eq. (3.1). The bunches travel along the beam pipe executing betatron oscillations with frequencies modified by resistive wall effects. The amplitude of all oscillations are the same and are normalized to unity. Values of all the parameters are taken from Table I. Results obtained using $I=30$ mA are also shown for comparison.

TABLE II. Normalized steady-state displacement $\xi(t,\infty)$ given by the various approximate solutions are compared to the exact result.

Approximation used	$\xi(t,\infty)$
Exact	-0.7373
Third order	-0.7435
Second order	-0.7438
First order	-0.7528
Dominant mode	-0.7462
Numerical integration	-0.7439

the dominant mode solution [Eq. (3.14)] and the first- [the first line in Eq. (3.13)], second- [Eq. (3.14)], and third-order solutions. The third-order solution has not been given in this paper, as it is quite complicated. A comparison of their results for the steady-state value is given in Table II. As expected, the third-order solution is closest to the exact result. However, both the second-order solution and the numerical integration of the equations of motion are seen to be quite adequate (for the parameter values given in Table I).

F. Transient solution

Figure 3 shows that the transverse bunch displacement approaches a steady state monotonically from below. This behavior is typical for parameters considered for rf-based FEL's. Figure 4 depicts a situation when parameters are beyond this typical range. The transverse displacement will exhibit oscillations before reaching a steady state. These oscillations represent a transient state and will be discussed briefly in this section.

A rough estimate as to when the transient state becomes important can be given as follows. It is seen from Eq. (3.13) that the transient amplitude of the Kth bunch is given by $\Omega_K^2 t/2\omega_0$ where

$$\Omega_K^2=\sum_{j=1}^{K}\sum_{i=1}^{\infty}G_i\exp(-j\Delta/T_i)\ . \tag{3.20}$$

Since Ω_K^2 is less than Ω^2 for all K, a general condition for the transient state to become important is

$$\frac{\Omega^2 t}{2\omega_0}=\frac{\Omega^2 z}{2\omega_0 c}\geq 1\ . \tag{3.21}$$

For the parameter values in Table I, this quantity has a value of 0.13. Equation (3.21) shows that the transient state becomes important when the length of wiggler increases. A more detailed analysis of the transient is being carried out and the results will be presented in a forthcoming paper.

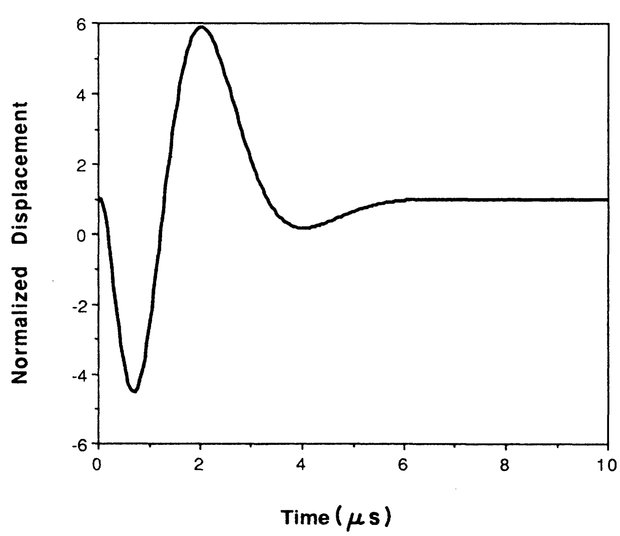

FIG. 4. An extreme situation where the transient state dominates the steady state. The parameter values used are $z=600$ m, $I=4.3$ A, and $\Delta=0.46$ ns. The other parameters have values given in Table I.

IV. CONCLUSIONS

The transverse motion of a beam traversing a narrow beam pipe is modified by resistive wall effects. Depending on the ratio of strengths of the focusing force due to the alternating wiggler field and the resistive wall effect, the effect on the steady state ranges from a modification of the betatron oscillation to a growth in transverse displacement with the length of pipe. The strength of the resistive wall effect depends only on the pipe radius b (increasing as $1/b^2$), but is independent of the thickness (τ) and conductivity (σ) of the pipe. However, τ and σ affect the time needed to attain the steady state. A transient state preceding the steady state can become important for parameter values beyond those considered here. A criterion for determining when the transient state becomes important has been derived.

ACKNOWLEDGMENTS

This work was supported and funded by the U. S. Department of Defense, Army Strategic Defense Command, under the auspices of the U. S. Department of Energy. One of us (G.R.) was supported by the Department of Energy, Contract No. DESA05-80ER10666. We would like to thank Dr. R. L. Gluckstern, Dr. F. Neri, Dr. P. J. Channell, Dr. V. K. Neil, and Dr. R. K. Cooper for useful discussions. One of us (G.R.) would also like to thank Dr. A. Dragt for his support and encouragement and AT-6, Los Alamos National Laboratory, for their warm hospitality.

APPENDIX A

This appendix describes the derivation of a complete solution of Eq. (3.1):

$$\frac{d^2\xi(t,K)}{dt^2}+\omega_0^2\xi(t,K)=\sum_{i=1}^{\infty}G_i\sum_{l=0}^{K-1}\exp[-(K-l)\Delta/T_i]\xi(t,l)\ . \tag{A1}$$

Equation (A1) can be rewritten as

$$\frac{d^2\xi(t,K)}{dt^2}+\omega_0^2\xi(t,K)=\sum_{i=1}^{\infty} G_i \sum_{l=0}^{K-1} s_{i,K-l}\xi(t,l) \ , \tag{A2}$$

where

$$s_{i,m}=\exp(-m\Delta/T_i) \ . \tag{A3}$$

Assuming that the initial transverse velocity of all the bunches is zero, Eq. (A2) can be Laplace transformed to

$$\frac{(y^2+\omega_0^2)}{y}\bar{\xi}(y,K)-\sum_{i=1}^{\infty}\frac{G_i}{y}\sum_{l=0}^{K-1} s_{i,K-l}\bar{\xi}(y,l)=\xi(0,K) \ , \tag{A4}$$

where

$$\bar{\xi}(y,K)=\int_0^{\infty} dt \exp(-yt)\xi(t,K) \ . \tag{A5}$$

Noting the similarity between Eq. (24) in Ref. 5 and Eq. (A4), the above equation can be written in matrix form as

$$A\bar{\xi}(y)=\xi(0) \ , \tag{A6}$$

where

$$A=\frac{(y^2+\omega_0^2)}{y}I-\frac{1}{y}\sum_{i=1}^{\infty} G_i S_i \ . \tag{A7}$$

Here I is the identity matrix and S_i is given by

$$S_i=\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ s_{i,1} & 0 & 0 & 0 & \cdots \\ s_{i,2} & s_{i,1} & 0 & 0 & \cdots \\ s_{i,3} & s_{i,2} & s_{i,1} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \end{bmatrix} \ . \tag{A8}$$

The matrix A has to be inverted to find $\bar{\xi}(y)$:

$$A^{-1}=\frac{y}{(y^2+\omega_0^2)}\left[I-\frac{1}{(y^2+\omega_0^2)}\sum_{i=1}^{\infty} G_i S_i\right]^{-1} \ .$$

Expressing this as a power series,

$$A^{-1}=\sum_{n=0}^{\infty}\frac{y}{(y^2+\omega^2)^{n+1}}\left[\sum_{i=1}^{\infty} G_i S_i\right]^n \ . \tag{A9}$$

The solution for $\bar{\xi}(y)$ can now be written as

$$\bar{\xi}(y)=\sum_{n=0}^{\infty}\frac{y}{(y^2+\omega^2)^{n+1}}\left[\sum_{i=1}^{\infty} G_i S_i\right]^n \xi(0) \ . \tag{A10}$$

To obtain $\xi(t)$, an inverse Laplace transform is performed:

$$\xi(t)=\sum_{n=0}^{\infty}\left[\sum_{i=1}^{\infty} G_i S_i\right]^n \xi(0)\frac{1}{2\pi i}\times\int_{\alpha-i\infty}^{\alpha+i\infty} dy\frac{y}{(y^2+\omega_0^2)^{n+1}}\exp(yt) \ , \tag{A11}$$

where α is an arbitrary positive constant. It can be shown that

$$\frac{1}{2\pi i}\int_{\alpha-i\infty}^{\alpha+i\infty} dy\frac{y}{(y^2+\omega_0^2)^{n+1}}\exp(yt) = \frac{\pi^{1/2}}{2^{n+(1/2)}\omega_0^{2n}n!}(\omega_0 t)^{n+(1/2)}J_{n-(1/2)}(\omega_0 t) \equiv F(n) \ , \tag{A12}$$

which implies

$$\xi(t)=\sum_{n=0}^{\infty} F(n)\left[\sum_{i=1}^{\infty} G_i S_i\right]^n \xi(0) \ . \tag{A13}$$

This can be rewritten as

$$\xi(t)=\sum_{n=0}^{\infty} F(n)\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_n=1}^{\infty} G_{i_1}G_{i_2}\cdots G_{i_n}S_{i_1}S_{i_2}\cdots S_{i_n}\xi(0) \ . \tag{A14}$$

Carrying out the matrix multiplication,

$$\xi(t,K)=\sum_{K'=0}^{K}\xi(0,K')\sum_{n=0}^{\infty} F(n)\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_n=1}^{\infty} G_{i_1}G_{i_2}\cdots G_{i_n}(S_{i_1}S_{i_2}\cdots S_{i_n})_{KK'} \ . \tag{A15}$$

Using Eq. (B14) and

$$(S_{i_1}S_{i_2}\cdots S_{i_n})_{KK'}=\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty}\cdots\sum_{j_{n-1}=1}^{\infty}(S_{i_1})_{Kj_1}(S_{i_2})_{j_1j_2}\cdots(S_{i_n})_{j_{n-1}K'} \ , \tag{A16}$$

Eq. (A15) can be written as

$$\begin{aligned}\xi(t,K)=&\sum_{K'=0}^{K}\xi(0,K')\sum_{n=0}^{\infty} F(n)\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_n=1}^{\infty} G_{i_1}G_{i_2}\cdots G_{i_n}\\ &\times\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty}\cdots\sum_{j_{n-1}=1}^{\infty} C^1_{K-j_1-1}C^1_{j_1-j_2-1}\cdots C^1_{j_{n-2}-j_{n-1}-1}\\ &\times C^1_{j_{n-1}-K'-1}\exp[-(K-j_1)\Delta/T_{i_1}]\exp[-(j_1-j_2)\Delta/T_{i_2}]\cdots\exp[-(j_{n-2}-j_{n-1})\Delta/T_{i_{n-1}}]\\ &\times\exp[-(j_{n-1}-K')\Delta/T_{i_n}] \ .\end{aligned} \tag{A17}$$

After further manipulation using the properties of C_l^n and changing the running index from K' to $m=K-K'$, the above equation finally reduces to

$$\tilde{\xi}(t,K)=\sum_{m=1}^{K}\xi(0,K-m)\sum_{n=1}^{m}F(n)\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_n=1}^{\infty}G_{i_1}G_{i_2}\cdots G_{i_n}\sum_{m_1=1}^{m-1}\sum_{m_2=1}^{m-m_1-1}\cdots\sum_{m_{n-1}=1}^{m-m_1-\cdots-m_{n-2}-1}$$
$$\times\exp(-m_1\Delta/T_{i_1})\exp(-m_2\Delta/T_{i_2})\cdots\exp(-m_{n-1}\Delta/T_{i_{n-1}})\exp[-(m-m_1-\cdots-m_{n-1})\Delta/T_{i_n}]\ , \quad (A18)$$

where

$$\tilde{\xi}(t,K)=\xi(t,K)-\xi(0,K)\cos(\omega_0 t)\ . \quad (A19)$$

Equation (A19) is a complete solution of Eq. (3.1).

APPENDIX B

An expression for the matrix elements of S_i^n is derived in this appendix. We drop the subscript i in the following discussion. Using the definition of S given in Eq. (A8), the matrix elements of S can be expressed as

$$S_{rs}=\sum_{m_1=1}^{\infty}s_{m_1}\delta_{r-s,m_1}\ . \quad (B1)$$

It can be shown that

$$(S^n)_{rs}=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}s_{m_1}s_{m_2}\cdots s_{m_n}\delta_{r-s,m_1+m_2+\cdots+m_n}\ . \quad (B2)$$

Using Eq. (A3),

$$(S^n)_{rs}=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\exp[-\Delta(m_1+m_2+\cdots+m_n)/T]\delta_{r-s,m_1+m_2+\cdots+m_n}\ . \quad (B3)$$

Let

$$P_s(r)=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\exp\left[-\Delta\sum_{i=1}^{n}m_i/T\right]\times\delta\left[r-s-\sum_{i=1}^{n}m_i\right]\ . \quad (B4)$$

Then

$$(S^n)_{rs}=\int_{r-\epsilon}^{r+\epsilon}dr P_s(r)\ . \quad (B5)$$

$P_s(r)$ is evaluated by Laplace transforming it:

$$\bar{P}_s(y)=\int_0^{\infty}dr\, P_s(r)\exp(-yr)\ . \quad (B6)$$

The δ function can be integrated over to give

$$\bar{P}_s(y)=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\cdots\sum_{m_n=1}^{\infty}\exp\left[-\Delta\sum_{i=1}^{n}m_i/T\right]\times\exp\left[-y\left[s+\sum_{i=1}^{n}m_i\right]\right]\ . \quad (B7)$$

This can be reexpressed as

$$\bar{P}_s(y)=\exp(-ys)\left\{\sum_{m=1}^{\infty}\exp\left[-m\left[y+\frac{\Delta}{T}\right]\right]\right\}^n\ . \quad (B8)$$

The infinite geometric series can be summed to give

$$\bar{P}_s(y)=\frac{\exp[-y(s+n)]\exp(-n\Delta/T)}{\{1-\exp[-(y+\Delta/T)]\}^n}\ . \quad (B9)$$

Since $y+\Delta/T$ is greater than zero,

$$\frac{1}{\{1-\exp[-(y+\Delta/T)]\}^n}=\sum_{l=0}^{\infty}C_l^n\exp\left[-l\left[y+\frac{\Delta}{T}\right]\right]\ , \quad (B10)$$

where

$$C_0^0=1\ ,\qquad C_l^n=(-1)^l\frac{(-n)(-n-1)\cdots(-n-l+1)}{l!}\ . \quad (B11)$$

Using this in Eq. (B9),

$$\bar{P}_s(y)=\sum_{l=0}^{\infty}C_l^n\exp[-y(s+n+l)]\exp[-\Delta(n+l)/T]\ . \quad (B12)$$

Comparing Eqs. (B4) and (B7), which are Laplace transforms of one another, the Laplace transform of the above equation is immediately seen to be

$$P_s(r)=\sum_{l=0}^{\infty}C_l^n\exp[-\Delta(n+l)/T]\delta(r-s-n-l)\ . \quad (B13)$$

Using Eq. (B5) and restoring the index i,

$$(S_i^n)_{rs}=\begin{cases}C_{r-s-n}^n\exp[-\Delta(r-s)/T_i] & \text{if } r-s\ge n\\ 0 & \text{otherwise}\ .\end{cases} \quad (B14)$$

APPENDIX C

In this appendix, $\xi(t,K)$ is shown to approach a steady state when $K\to\infty$. An expression for the steady-state value of the transverse displacement $\xi(t,\infty)$ is also derived.

Assuming that $\xi(0,j)=d$ for all j and interchanging the summations over m and n in Eq. (3.3), we get

$$\tilde{\xi}(t,K)=d\sum_{n=1}^{K}F(n)\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_n=1}^{\infty}G_{i_1}G_{i_2}\cdots G_{i_n}\sum_{m=n}^{K}\sum_{m_1=1}^{m-1}\sum_{m_2=1}^{m-m_1-1}\cdots\sum_{m_{n-1}=1}^{m-m_1-\cdots-m_{n-2}-1}\exp(-m_1\Delta/T_{i_1})$$
$$\times\exp(-m_2\Delta/T_{i_2})\cdots\exp(-m_{n-1}\Delta/T_{i_{n-1}})\exp[-(m-m_1-\cdots-m_{n-1})\Delta/T_{i_n}]\ . \quad \text{(C1)}$$

Letting $K\to\infty$ and setting $m=m_n$,

$$\tilde{\xi}(t,\infty)=d\sum_{n=1}^{\infty}F(n)\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{\infty}\cdots\sum_{i_n=1}^{\infty}G_{i_1}G_{i_2}\cdots G_{i_n}\sum_{m_n=n}^{\infty}\sum_{m_1=1}^{m_n-1}\sum_{m_2=1}^{m_n-m_1-1}\cdots\sum_{m_{n-1}=1}^{m_n-m_1-\cdots-m_{n-2}-1}$$
$$\times\exp(-m_1\Delta/T_{i_1})\exp(-m_2\Delta/T_{i_2})\cdots\exp(-m_{n-1}\Delta/T_{i_{n-1}})\exp[-(m_n-m_1-\cdots-m_{n-1})\Delta/T_{i_n}]\ . \quad \text{(C2)}$$

In this limit ($K=\infty$), the above equation can be reduced to a much simpler expression. After considerable manipulation using standard results on multiplication of power series, the final expression turns out to be

$$\tilde{\xi}(t,\infty)=d\sum_{n=1}^{\infty}F(n)(\Omega^2)^n\ , \quad \text{(C3)}$$

where

$$\Omega^2=\sum_{i=1}^{\infty}G_i\sum_{j=1}^{\infty}\exp(-j\Delta/T_i)\ . \quad \text{(C4)}$$

Using Eqs. (3.4) and (3.5) this reduces to

$$\xi(t,\infty)=d\sum_{n=0}^{\infty}\left[\frac{\Omega^2 t}{2\omega_0}\right]^n\left[\frac{\pi\omega_0 t}{2}\right]^{1/2}\frac{1}{n!}J_{n-1/2}(\omega_0 t)\ . \quad \text{(C5)}$$

Let

$$\omega_0 t=x\ ,\qquad \Omega t=y\ . \quad \text{(C6)}$$

Then

$$\xi(t,\infty)=d\sum_{n=0}^{\infty}\left[\frac{y^2}{2x}\right]^n\left[\frac{\pi x}{2}\right]^{1/2}\frac{1}{n!}J_{n-1/2}(x)\ . \quad \text{(C7)}$$

Since

$$J_{n-1/2}(x)=\sum_{m=0}^{\infty}\frac{(-1)^m(x/2)^{2m+n-1/2}}{m!(m+n-1/2)!}\ , \quad \text{(C8)}$$

Eq. (C5) becomes

$$\xi(t,\infty)=d\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^m\pi^{1/2}y^{2n}x^{2m}}{2^{2m+2n}(m+n-1/2)!m!n!}\ . \quad \text{(C9)}$$

The above equation can be further manipulated to give

$$\xi(t,\infty)=d\sum_{l=0}^{\infty}\frac{1}{(2l)!}\sum_{m=0}^{\infty}\frac{l!(y^2)^{l-m}(-x^2)^m}{m!(l-m)!}\ , \quad \text{(C10)}$$

where $l=n+m$. Noticing that

$$\sum_{m=0}^{\infty}\frac{l!(y^2)^{l-m}(-x^2)^m}{m!(l-m)!}=(y^2-x^2)^l\ , \quad \text{(C11)}$$

Eq. (C10) can be written as

$$\xi(t,\infty)=d\sum_{l=0}^{\infty}\frac{(y^2-x^2)^l}{(2l)!}\ . \quad \text{(C12)}$$

Using the standard power-series expansion for $\cosh(x)$ and Eq. (C6), the final expression for $\xi(t,\infty)$ is found to be

$$\xi(t,\infty)=d\cosh[(\Omega^2-\omega_0^2)^{1/2}t]\ . \quad \text{(C13)}$$

If $\omega_0^2>\Omega^2$,

$$\xi(t,\infty)=d\cos[(\omega_0^2-\Omega^2)^{1/2}t]\ . \quad \text{(C14)}$$

Thus the transverse displacement approaches a finite limiting value (independent of K) as K becomes large. This proves that a steady-state solution exists and its value is given by either Eq. (C13) or (C14).

APPENDIX D

This appendix gives the derivation of a simplified solution of Eq. (3.1) when the first mode dominates. Denoting the transverse displacement of the Kth bunch under the influence of the first mode by $\xi_1(t,K)$, Eq. (A17) reduces in this case to

$$\xi_1(t,K)=\sum_{K'=0}^{K}\xi_1(0,K')\sum_{n=0}^{\infty}F(n)G_1^n\exp[-(K-K')\Delta/T_1]\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty}\cdots\sum_{j_{n-1}=1}^{\infty}C_{K-j_1-1}^{1}C_{j_1-j_2-1}^{1}\cdots C_{j_{n-1}-K'-1}^{1}\ . \quad \text{(D1)}$$

This can be simplified using the sum relation

$$\sum_{j_1=1}^{\infty}\sum_{j_2=1}^{\infty}\cdots\sum_{j_{n-1}=1}^{\infty} C^r_{K-j_1-1}C^r_{j_1-j_2-1}\cdots C^r_{j_{n-1}-K'-1} = C^{nr}_{K-K'} \ . \quad \text{(D2)}$$

This relation can be proved as follows. Start with the identity [cf. Eq. (B10)]

$$\frac{1}{\{1-\exp[-(y+\Delta/T)]\}^{2r}} = \sum_{l=0}^{\infty} C_l^{2r}\exp\left[-l\left[y+\frac{\Delta}{T}\right]\right] . \quad \text{(D3)}$$

The left-hand side can be written as

$$\sum_{l=0}^{\infty} C_l^r\exp\left[-l\left[y+\frac{\Delta}{T}\right]\right]\sum_{k=0}^{\infty} C_k^r\exp\left[-k\left[y+\frac{\Delta}{T}\right]\right] ,$$

but

$$\sum_{l=0}^{\infty} a_l x^l \sum_{k=0}^{\infty} a_k x^k = \sum_{l=0}^{\infty}\sum_{k=0}^{l} a_{l-k}a_k x^l \ . \quad \text{(D4)}$$

Therefore the left-hand side becomes

$$\sum_{l=0}^{\infty}\sum_{k=0}^{l} C^r_{l-k}C^r_k\exp(-l\Delta/T) \ . \quad \text{(D5)}$$

Comparing it with the right-hand side of Eq. (D3),

$$\sum_{k=0}^{l} C^r_{l-k}C^r_k = C_l^{2r} \ . \quad \text{(D6)}$$

After some manipulation, this can be rewritten as

$$\sum_{j_1=0}^{\infty} C^r_{K-j_1}C^r_{j_1-K'} = C^{2r}_{K-K'} \ . \quad \text{(D7)}$$

This sum relation can be iterated to give Eq. (D2).

Using Eq. (D2), Eq. (D1) can be reduced to

$$\xi_1(t,K) = \sum_{K'=0}^{K}\xi_1(0,K')\sum_{n=0}^{\infty}F(n)G_1^nC^n_{K-K'-n} \times\exp[-(K-K')\Delta/T_1] \ . \quad \text{(D8)}$$

Changing the running index from $K-K'$ to k and using Eq. (3.5), we get

$$\xi_1(t,K) = \sum_{k=0}^{K}\xi_1(0,K-k)\exp(-k\Delta/T_1) \times\sum_{n=0}^{k}\frac{G_1^nC^n_{k-n}\pi^{1/2}}{2^{n+1/2}\omega_0^{2n}n!}(\omega_0 t)^{n+1/2} \times J_{n-1/2}(\omega_0 t) \ . \quad \text{(D9)}$$

The fact that C^n_{k-n} is zero for $n>k$ has been used to obtain the above expression.

Under the assumption $\xi_1(0,j)=d$ for all j, $\xi_1(t,K)$ can be shown to approach a steady-state value $\xi_1(t,\infty)$, i.e., a value independent of K. The easiest way to show this is to go through the derivation given in Appendix C but with only the first mode present. It is seen that the derivation remains valid even in this case once Ω^2 has been replaced by

$$\Omega_1^2 = G_1\sum_{j=1}^{\infty}\exp(-j\Delta/T_1) \ . \quad \text{(D10)}$$

The steady-state value $\xi_1(t,\infty)$ is given as (when $\omega_0^2>\Omega^2$)

$$\xi_1(t,\infty) = d\cos[(\omega_0^2-\Omega_1^2)^{1/2}t] \ . \quad \text{(D11)}$$

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