# Transient state of a bunched electron beam subject to resistive-wall instability

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(Received 13 June 1989)

This paper analyzes the transient state of a bunched electron beam traversing a narrow beam pipe under the influence of transverse resistive-wall effects. Because the electron beam in the wigglers of future free-electron lasers (FEL's) is required to travel in pipes with small diameters, this study is relevant in determining the beam stability in these FEL's. The analysis is restricted to the case in which the first mode dominates. First, an integral representation of the single-mode solution is obtained. Second, the transient-state solution is obtained for the case in which the focusing force dominates the resistive-wall force. The maximum transient amplitude is derived for the case in which a single bunch is off-axis. Third, this result is used to compute the maximum transient amplitudes when the bunches enter the system with random and with equal initial displacements, respectively. These analytic expressions of maximum transient amplitudes are useful in predicting when the transient could become significant. They are compared with earlier derivations and with numerical simulations. These expressions are also applied to various proposed FEL's. Results showed that the transverse resistive-wall instability could not pose a problem in the operation of these FEL's.

## I. INTRODUCTION

The resistive-wall effects on the transverse beam dynamics of a bunched electron beam traversing the wiggler of a free-electron laser (FEL) have been discussed earlier.[1,2] The equation of motion for a bunched electron beam was solved exactly and a steady-state solution was obtained. However, the transient state was discussed only briefly. A more detailed discussion of the transient state is required because the transient state could cause large excursions in beam position and lead to beam loss even if the steady state is well behaved. This paper is an attempt to fill this gap in our understanding of the transverse resistive-wall instability.

We restrict ourselves to the case in which the first mode dominates. The first mode is the first term in an expansion of the field induced by the beam.[2] It has a decay time usually referred to as diffusion time. This situation usually occurs when the beam-pipe thickness is small. In Sec. II, the transverse displacement of the $K$th bunch is obtained. This derivation is a generalization of the earlier one[1,2] such that the initial velocity of bunches is no longer restricted to zero. An integral representation of this solution is derived. In Sec. III, this integral representation is used to obtain analytic formulas characterizing the transient states for three different cases—"single pulse," bunches coming in with random initial displacements, and bunches coming in with equal initial displacements. Single pulse is defined as the case when only the first bunch has an initial displacement. Results are compared with numerical simulations and are also applied to various proposed FEL's to determine whether the transient state could lead to beam breakup. Section IV contains a summary of our results.

## II. SINGLE-MODE SOLUTION FOR NONZERO INITIAL VELOCITY

In this section, we obtain an analytic solution for the transverse displacement of a bunched beam in a pipe. The bunches composing the beam are allowed to have nonzero values for both initial displacements and velocities in the transverse direction. An integral representation of this solution is derived. Using this representation, a steady-state solution is shown to exist as the bunch number tends to infinity. When the initial transverse velocities are set to zero, this steady-state solution agrees with the one derived earlier by a more direct method.[2]

The beam is a series of bunches traveling with speed $v$ (in the $z$ direction) in a pipe of inner radius $b$, outer radius $d$, thickness $\tau(=d-b)$, and conductivity $\sigma$. Each bunch carries a charge $q$, and there is a fixed time interval $\Delta$ between any two bunches. The transverse displacement from the axis of the $K$th bunch is denoted by $\xi(z=ct,K)$. Note that $t=0$ denotes the time when the $K$th bunch enters the pipe $(z=0)$.

The equation of motion for $\xi(t,K)$ has been derived earlier and found to be[2]

$$\frac{d^2\xi(t,K)}{dt^2}+\omega_0^2\xi(t,K)=\sum_{i=1}^{\infty} G_i \sum_{l=0}^{K-1} \exp[-(K-l)\Delta/T_i]\xi(t,l) , \quad (2.1)$$

 

where

$$G_i = \frac{8eqv}{m\gamma c^2 b^2 C_i T_i} \ , \tag{2.2}$$

$$T_i = \frac{4\pi\sigma b^2}{c^2 y_i^2} \ , \tag{2.3}$$

$$C_i = y_i^2 \left[ \left[ \frac{J_2(y_i)}{J_0(y_i d/b)} \right]^2 - 1 \right] , \tag{2.4}$$

and $y_i$'s are the zeros of the function

$$J_0(yd/b)N_2(y) - J_2(y)N_0(yd/b) \ .$$

Here, $J_i$'s and $N_i$'s are Bessel functions of the first and second kind, respectively. The quantity $\omega_0$ in the above expressions is the frequency of the slow betatron motion in the wiggler, and $\gamma$ is the usual relativistic factor. The sum over $l$ in Eq. (2.1) represents a sum of the interactions between the $K$th bunch and the resistive-wall wake fields of all bunches ahead of it. The summation over $i$ is a summation over an infinite number of modes that are due to resistive-wall effect.

In many situations, only the first mode dominates. For example, this happens when the thickness of the pipe $\tau$ is small compared to the radius $b$. Henceforth, we restrict ourselves to this single-mode case. Equation (2.1) then reduces to

$$\frac{d^2\xi(t,K)}{dt^2} + \omega_0^2 \xi(t,K) = G \sum_{l=0}^{K-1} \exp[-(K-l)\Delta/T]\xi(t,l) \ , \tag{2.5}$$

where the subscript $i=1$ has been dropped and

$$G = \frac{eqv}{m\gamma\pi\sigma\tau b^3}$$

and the diffusion time

$$T = \frac{2\pi\sigma b\tau}{c^2} \ . \tag{2.6}$$

Equation (2.5) can be rewritten as two first-order differential equations:

$$\frac{d\xi(t,K)}{dt} = \eta(t,K) \tag{2.7a}$$

and

$$\frac{d\eta(t,K)}{dt} + \omega_0^2 \xi(t,K) = G \sum_{l=0}^{K-1} s_{K-l}\xi(t,l) \ , \tag{2.7b}$$

where

$$s_m = \exp(-m\Delta/T) \ . \tag{2.8}$$

Equations (2.7a) and (2.7b) can be Laplace transformed to

$$y\bar{\xi}(y,K) - \bar{\eta}(y,K) = \xi(0,K) \tag{2.9}$$

and

$$\omega_0^2 \bar{\xi}(y,K) - \frac{G}{y} \sum_{l=0}^{K-1} s_{K-l} \bar{\xi}(y,l) + y\bar{\eta}(y,K) = \eta(0,K) \ , \tag{2.10}$$

where

$$\bar{\xi}(y,K) = \int_0^\infty dt \exp(-yt)\xi(t,K)$$

and

$$\bar{\eta}(y,K) = \int_0^\infty dt \exp(-yt)\eta(t,K) \ . \tag{2.11}$$

Solving Eqs. (2.9) and (2.10) for $\bar{\xi}(y)$, the following matrix equation is obtained:

$$A\bar{\xi}(y) = y\xi(0) + \eta(0) \ , \tag{2.12}$$

where

$$A = (y^2 + \omega_0^2)I - GS \ . \tag{2.13}$$

Here, $I$ is the identity matrix and $S$ is given by

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ s_1 & 0 & 0 & 0 & \cdots \\ s_2 & s_1 & 0 & 0 & \cdots \\ s_3 & s_2 & s_1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} . \tag{2.14}$$

Inverting the matrix $A$, the solution for $\bar{\xi}(y)$ is found to be

$$\bar{\xi}(y) = \sum_{n=0}^{\infty} \frac{1}{(y^2+\omega^2)^{n+1}} (GS)^n [y\xi(0) + \eta(0)] \ . \tag{2.15}$$

An inverse Laplace transform is performed to obtain $\xi(t)$. After considerable manipulation, the final expression for $\xi(t,K)$ is found to be

$$\xi(t,K) = \sum_{k=0}^{K} \exp(-k\Delta/T) \sum_{n=0}^{\infty} \left[ \frac{\pi\omega_0 t}{2} \right]^{1/2} \left[ \frac{Gt}{2\omega_0} \right]^n \frac{1}{n!} \binom{k-1}{k-n} \times \left[ J_{n-1/2}(\omega_0 t)\xi(0,K-k) + \frac{1}{\omega_0} J_{n+1/2}(\omega_0 t)\eta(0,K-k) \right] . \tag{2.16}$$

This represents a complete analytic solution for $\xi(t,K)$. This result can be generalized to include all modes. However, it is not easy to generalize the other results that follow. Hence, the analysis in this paper has been restricted to a single-mode case.

To obtain a transient-state solution (which is the ultimate goal of this paper), the result given in Eq. (2.16) is transformed into an integral representation in Appendix A. The final result [cf. Eqs. (A1), (A13), and (A14)] is

$$\xi(t,K)=\sum_{k=0}^{K}(-1)^k\exp(-k\Delta/T)[R_1(t,k)\xi(0,K-k)+R_2(t,k)\eta(0,K-k)]\ , \tag{2.17}$$

where

$$R_1(t,k)=\frac{1}{2\pi i}\oint du\,\frac{1}{u^{2k+1}}\cos\left[\left[\omega_0^2+\frac{Gu^2}{1+u^2}\right]^{1/2}t\right] \tag{2.18}$$

and

$$R_2(t,k)=\frac{1}{2\pi i}\oint du\left[\omega_0^2+\frac{Gu^2}{1+u^2}\right]^{-1/2}\frac{1}{u^{2k+1}}\sin\left[\left[\omega_0^2+\frac{Gu^2}{1+u^2}\right]^{1/2}t\right]\ . \tag{2.19}$$

The contours in Eqs. (2.18) and (2.19) enclose the origin and $|u|<1$ everywhere on the contours. Equation (2.17) is now in a form that can be used to derive the transient-state solution.

Before proceeding with the derivation of the transient state, it is a useful exercise to investigate the steady-state solution of Eq. (2.17) [or equivalently Eq. (2.16)]. From our earlier work,[2] we already know the steady-state solution when the initial transverse velocity is zero for all bunches. Comparing the steady-state solution obtained using Eq. (2.17) with this earlier result would be a useful check on the validity of Eq. (2.17).

The steady-state solution when $\xi(0,j)=d_1$ and $\eta(0,j)=d_2$ for all $j$ can be obtained by letting $K\rightarrow\infty$:

$$\xi(t,\infty)=\frac{1}{2\pi i}\oint du\left\{d_1\cos\left[\left[\omega_0^2+\frac{Gu^2}{1+u^2}\right]^{1/2}t\right]+d_2\left[\omega_0^2+\frac{Gu^2}{1+u^2}\right]^{-1/2}\sin\left[\left[\omega_0^2+\frac{Gu^2}{1+u^2}\right]^{1/2}t\right]\right\}\sum_{k=0}^{\infty}(-1)^m\frac{\exp(-k\Delta/T)}{u^{2k+1}}\ . \tag{2.20}$$

Summing the geometric series,

$$\xi(t,\infty)=\frac{1}{2\pi i}\oint du\left\{d_1\cos\left[\left[\omega_0^2+\frac{Gu^2}{1+u^2}\right]^{1/2}t\right]+d_2\left[\omega_0^2+\frac{Gu^2}{1+u^2}\right]^{-1/2}\sin\left[\left[\omega_0^2+\frac{Gu^2}{1+u^2}\right]^{1/2}t\right]\right\}\frac{u}{u^2+\exp(-\Delta/T)}\ . \tag{2.21}$$

The integral is evaluated as the sum of residues at poles $u=\pm i\exp(-\Delta/2T)$ to give

$$\xi(t,\infty)=d_1\cos[(\omega_0^2-\Omega^2)^{1/2}t]+d_2\sin[(\omega_0^2-\Omega^2)^{1/2}t](\omega_0^2-\Omega^2)^{-1/2}\ , \tag{2.22}$$

where

$$\Omega^2=\frac{G\exp(-\Delta/T)}{1-\exp(-\Delta/T)}=\frac{2eIv}{m\gamma c^2b^2}\ . \tag{2.23}$$

Setting $d_2$ equal to zero, it is seen that the above result agrees with that given in Eq. (3.16) of Ref. 2.

If $\Omega^2>\omega_0^2$, Eq. (2.22) transforms into

$$\xi(t,\infty)=d_1\cosh[(\Omega^2-\omega_0^2)^{1/2}t]+d_2\sinh[(\Omega^2-\omega_0^2)^{1/2}t](\Omega^2-\omega_0^2)^{-1/2}\ . \tag{2.24}$$

When the resistive-wall force equals the focusing force (i.e., when $\Omega^2=\omega_0^2$),

$$\xi(t,\infty)=d_1+d_2t\ . \tag{2.25}$$

## III. DERIVATION OF THE TRANSIENT-STATE SOLUTION

In this section, we study the transient state of Eq. (2.17) before a steady state is attained. The motivation for this study is as follows: When $\Omega^2\geq\omega_0^2$, the steady-state solution grows exponentially with the length of the pipe [cf. Eq. (2.24)] leading to beam breakup. Therefore, FEL's should not be operated in this regime. However, the possibility for an instability exists even if $\omega_0^2\geq\Omega^2$. Even though the steady state is well behaved in this case [cf. Eq. (2.22)], earlier studies of related problems[3,4] have shown that the transient state could have large transverse excursions leading to beam loss. An extreme example exhibiting this phenomenon is shown in Fig. 1.

To investigate this possibility, we undertake a systematic study of the transient state in this section. This study is carried out for the case in which $\eta(0,K)=0$. It is further assumed that $\omega_0^2\gg G$, an assumption valid in most practical situations. We analyze whether this transient state could pose a problem in some of the FEL's that have been proposed[5] (their relevant parameter values are given in Table I). All these FEL's have parameter values that satisfy the condition $\omega_0^2\gg G$. The goal is to develop criteria which determine when a significant transient exists.

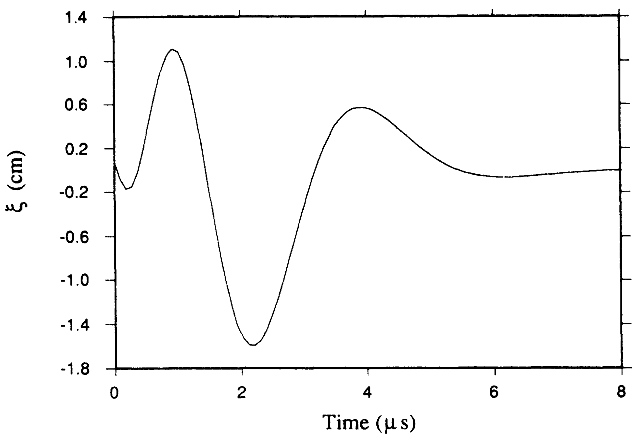


FIG. 1. Transverse displacement of the beam at the end of the wiggler for a hypothetical FEL as a function of time elapsed since the passage of the leading bunch. The transient state is seen to dominate the steady state. This hypothetical FEL has parameter values $I=4.3$ A, $\Delta=0.46$ ns, $\omega_0=1.5\times10^9\ \mathrm{s}^{-1}$, and pipe length=600 m. The order parameters have the same values as those given for the xuv FEL in Table I. All bunches come in with the same initial displacement of 1.0 mm.

In Sec. III A, the transient state is derived for a beam where the first bunch is displaced off axis by an amount $d$ and the subsequent bunches follow on axis. This case will be referred to as the single-pulse case. In Sec. III B, the single-pulse result is used to obtain the transient state for a beam where the bunches come in with random initial displacements. In Sec. III C, a similar analysis is carried out for bunches coming in with equal initial displacements.

### A. Transient state for a single pulse

In this subsection, the transient solution for a single pulse is derived. The result is compared with numerical simulations and with other results derived earlier.

Setting $\xi(0,0)=d$, $\xi(0,k)=0$ for all $k>1$, and $\eta(0,k)=0$ for all $k$ in Eq. (2.17), the integral representation of $\xi(t,K)$ for a single pulse is given as follows [cf. Eq. (2.17)]:

$$\xi(t,K)=d(-1)^K\exp(-K\Delta/T)\frac{1}{2\pi i}\times\oint du\frac{1}{u^{2k+1}}\cos\left[\left[\omega_0^2+\frac{Gu^2}{1+u^2}\right]^{1/2}t\right]. \quad (3.1)$$

A saddle-point evaluation of the integral has been carried out in Appendix B and the final result is given below [cf. Eq. B12)]:

$$\xi(t,K)\cong\frac{d}{K\sqrt{8\pi}}\left[\frac{2KGt}{\omega_0}\right]^{1/4}\exp\left[-\frac{K\Delta}{T}+\left[\frac{KGt}{\omega_0}\right]^{1/2}\right]\times\mathrm{Re}\left\{\exp\left[i\omega_0 t-i\frac{\pi}{8}+i\frac{Gt}{4\omega_0}-i\left[\frac{KGt}{\omega_0}\right]^{1/2}\right]\right\}. \quad (3.2)$$

For a fixed $t$ (or equivalently, for a fixed location $z=vt$ along the pipe), $\xi(t,K)$ reaches a maximum when

$$K_m^{1/2}\cong\left[\frac{Gt}{4\omega_0}\right]^{1/2}\frac{T}{2\Delta}+\left[\left[\frac{Gt}{4\omega_0}\right]-\frac{3\Delta}{T}\right]^{1/2}\frac{T}{2\Delta}, \quad (3.3)$$

with a normalized (setting $d=1$) maximum transient displacement for a single pulse of

$$\xi_m(t,K_m)\cong\frac{1}{K_m\sqrt{8\pi}}\left[\frac{2K_mGt}{\omega_0}\right]^{1/4}\times\exp\left[-\frac{K_m\Delta}{T}+\left[\frac{K_mGt}{\omega_0}\right]^{1/2}\right]. \quad (3.4)$$

From Eq. (3.3), it is seen that the bunch number $K_m$ at which the maximum is reached decreases as $Gt/4\omega_0$ decreases. The equation breaks down for

$$\left[\frac{Gt}{4\omega_0}\right]<\frac{3\Delta}{T}. \quad (3.5)$$

TABLE I. Proposed parameters for various FEL's.

| Parameter | Compact FEL | Compact xuv FEL | FEL 1 | FEL 2 | xuv FEL |
|---|---|---|---|---|---|
| $b$ (cm) | 0.03 | 0.10 | 1.20 | 0.50 | 0.18 |
| $d$ (cm) | 0.033 | 0.11 | 1.40 | 0.70 | 0.198 |
| $\sigma$ ($10^{16}\ \mathrm{s}^{-1}$) | 2.0 | 2.0 | 2.0 | 2.0 | 2.0 |
| $B$ (T) | 4.5 | 3.0 | 0.36 | 0.447 | 0.75 |
| Pulse length (ms) | 0.01 | 0.3 | $\infty$ | $\infty$ | 300.0 |
| Energy (MeV) | 20.0 | 80.0 | 150.0 | 100.0 | 500.0 |
| $z$ (m) | 0.1 | 1.5 | 72.0 | 14.9 | 8.0 |
| $\Delta$ (ns) | 15.0 | 68.0 | 18.5 | 18.5 | 6.8 |
| $I$ (mA) | 500.0 | 300.0 | 625.0 | 310.0 | 300.0 |
| $\omega_0$ ($10^8\ \mathrm{s}^{-1}$) | 101.1 | 16.9 | 1.1 | 2.0 | 0.9 |
| $G$ ($10^{13}\ \mathrm{s}^{-2}$) | $1.6\times10^5$ | 905.2 | 0.007 | 0.071 | 1.4 |
| $T$ ($\mu$s) | 0.01 | 0.14 | 34.56 | 12.13 | 0.46 |

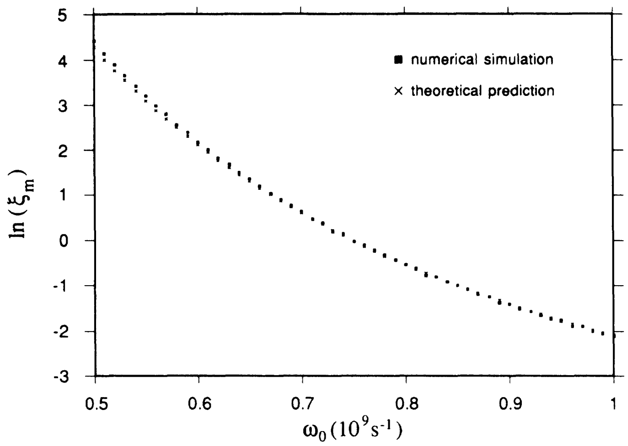


FIG. 2. Variation of the logarithm of normalized maximum transverse displacement $\xi_m$ as a function of the betatron frequency $\omega_0$ for the case of a single pulse. The results obtained using the analytical formula [Eq. (3.4)] are compared with those obtained using numerical simulations.

This suggests that there is no maximum to be found and that the transient state disappears for these parameter values when Eq. (3.5) is satisfied. Numerical simulations bear this result out. Five proposed FEL's (whose relevant parameter values are given in Table I) were studied numerically. All five have parameter values that satisfy Eq. (3.5) and none of them exhibits a significant transient.

To exhibit a case where the transient is present, a hypothetical FEL with extreme values for some parameters is considered. This FEL has $I=4.3$ A and pipe length=600 m. The other parameters have the same values as those given for xuv FEL (cf. Table I). Figure 2 shows a comparison of maximum transient displacement between numerical simulations and Eq. (3.4) for different values of the focusing strength (i.e., $\omega_0$). The agreement between the two is seen to be good for large $\omega_0$'s and starts to deteriorate slightly for small $\omega_0$'s. The deterioration is expected because Eq. (3.4) is valid only for focusing forces large compared to the resistive-wall forces.

Finally we compare our results with those derived earlier. Neil and Whittum[6] have obtained the following results for a single-pulse case:

$$K_m \cong \frac{\omega_1^2 z T}{4\omega_0 c \Delta} \tag{3.6}$$

and

$$\xi_m(t,K_m) \cong \exp\left[\frac{\omega_1^2 z}{4\omega_0 c}\right], \tag{3.7}$$

where

$$\omega_1^2 = \frac{2I(\mathrm{kA})c^2}{17\gamma b^2}, \tag{3.8}$$

$I$ is the average current, 17 (kA)$=mc^3/e$, and $z=ct$. In obtaining these results, several assumptions were made and the nonexponential terms in Eq. (3.2) were neglected.

Our results are more general and include the nonexponential terms. These terms contribute significant corrections to the maximum transient amplitude and, hence, should not be neglected. When only the exponential in Eq. (3.2) is considered, Eqs. (3.3) and (3.4) are reduced to

$$K_m \cong \left[\frac{Gt}{2\omega_0}\right]\frac{T^2}{2\Delta^2} \tag{3.9}$$

and

$$\xi_m(t,K_m) \cong \exp\left[\left[\frac{Gt}{2\omega_0}\right]\frac{T}{2\Delta}\right]. \tag{3.10}$$

Replacing $v$ and $q/\Delta$ by $c$ and $I$, respectively, we get

$$K_m \cong \frac{eItT}{2m\gamma cb^2\omega_0\Delta} \tag{3.11}$$

and

$$\xi_m(t,K_m) \cong \exp\left[\frac{eIt}{2m\gamma cb^2\omega_0}\right]. \tag{3.12}$$

These results are identical to Eqs. (3.6) and (3.7).

Next, the above results are compared with the heuristic result obtained in our previous paper.[2] The transient state was found to become significant when [cf. Eq. (3.21) in Ref. 2 after being adapted to the single-mode case]

$$\frac{\Omega^2 t}{2\omega_0} \geq 1. \tag{3.13}$$

It can be put in a different form:

$$\frac{\Omega^2 t}{2\omega_0} \cong \left[\frac{Gt}{2\omega_0}\right]\frac{T}{\Delta}. \tag{3.14}$$

Therefore, the condition in Eq. (3.13) corresponds approximately to the argument of the exponential in Eq. (3.10) being greater than one.

### B. Transient state for a random initial displacement

In this subsection, we derive an expression for the root-mean-squared displacement when the bunches have random initial displacements and zero initial velocities. The result is compared with numerical simulations.

For this case, the solution for a single pulse in Eq. (3.2) can be rewritten in the following form:

$$\xi(t,K) \cong \xi_0[f(t,K)+f^*(t,K)], \tag{3.15}$$

where

$$f(t,K) \cong \frac{1}{K\sqrt{32\pi}}\left[\frac{2KGt}{\omega_0}\right]^{1/4}\exp\left[-\frac{K\Delta}{T}+\left[\frac{KGt}{\omega_0}\right]^{1/2}\right] \times\exp\left[i\omega_0 t - i\frac{\pi}{8}+i\frac{Gt}{4\omega_0}-i\left[\frac{KGt}{\omega_0}\right]^{1/2}\right], \tag{3.16}$$

and $f^*(t,K)$ is the complex conjugate of $f(t,K)$. The transverse displacement $\xi(t,K)$, when the bunches come in with random initial displacements, can be written as a linear superposition of solutions of this form.[7] Thus,

$$\xi(t,K) \cong \sum_{k=0}^{K} [f(t,k)+f^*(t,k)]\xi(0,K-k) . \quad (3.17)$$

Because the initial displacements of two successive bunches are uncorrelated and $f(t,k)$ oscillates only very slowly with $k$, the mean-squared value for $\xi(t,K)$ as $K$ tends to infinity is given in terms of the mean-squared initial displacement $\langle \xi_0^2 \rangle$ as

$$\langle \xi^2(t,\infty) \rangle \cong 4\langle \xi_0^2 \rangle \sum_{k=0}^{\infty} |f(t,k)|^2 . \quad (3.18)$$

The sum over bunch numbers can now be approximated by an integral:

$$\langle \xi^2(t,\infty) \rangle \cong 4\langle \xi_0^2 \rangle \int_0^{\infty} dk\, |f(t,k)|^2 . \quad (3.19)$$

Substituting Eq. (3.16) in the above equation,

$$\langle \xi^2(t,\infty) \rangle \cong A(t,k) \langle \xi_0^2 \rangle \int_0^{\infty} dk\, e^{r(t,k)} , \quad (3.20)$$

where

$$A(t,k) = \frac{1}{8k^2\pi} \left[ \frac{2kGt}{\omega_0} \right]^{1/2} \quad (3.21)$$

and

$$r(t,k) = -\frac{2k\Delta}{T} + 2\left[ \frac{kGt}{\omega_0} \right]^{1/2} . \quad (3.22)$$

The integral in Eq. (3.20) can be evaluated using the saddle-point method. The saddle point occurs at the same value of the bunch number as for the single-pulse case and, hence, is given by $k=K_m$ [cf. Eq. (3.3)]. Using the standard expression from saddle-point theory, we obtain

$$\langle \xi^2(t,\infty) \rangle \cong A(t,K_m) \langle \xi_0^2 \rangle \left[ \frac{2\pi}{-r''(t,K_m)} \right]^{1/2} \times \exp[r(t,K_m)] . \quad (3.23)$$

Evaluating this and comparing with Eq. (3.4), we get

$$\frac{\langle \xi^2(t,\infty) \rangle}{\langle \xi_0^2 \rangle} \cong 2\pi^{1/2} \left[ \frac{\omega_0}{Gt} \right]^{1/4} K_m^{3/4} |\xi_m(t,K_m)|^2 , \quad (3.24)$$

where $\xi_m(t,K_m)$ is the normalized maximum transient amplitude for the single-pulse case.

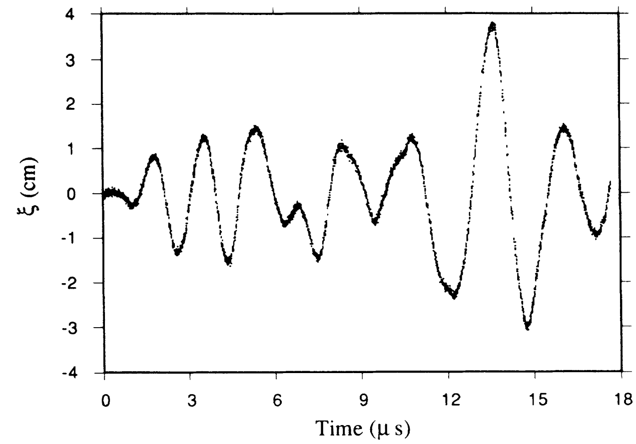


FIG. 3. Numerical simulation of the transverse displacement of the beam in the hypothetical FEL when the bunches come in with random initial displacements. The rms value of the initial displacements was 1 mm. The betatron frequency value is $9\times10^8$ s$^{-1}$.

This result has been compared with numerical simulations for our hypothetical FEL. Figure 3 shows the results of numerical simulation for $\omega_0 = 9\times10^8$ s$^{-1}$ with $\langle \xi_0^2 \rangle^{1/2} = 1$ mm. It gives a value of $\sim 1.2$ cm for $\langle \xi^2(t,\infty) \rangle^{1/2}$. From Eq. (3.24) and Fig. 2, we obtain a theoretical value of $\sim 1.3$ cm for $\langle \xi^2(t,\infty) \rangle^{1/2}$. Thus, there is good agreement between theory and simulation.

### C. Transient state for equal initial displacement

In this subsection, we derive an expression for the maximum transient displacement when all bunches are displaced off axis by an amount $d$ and have no transverse velocity, using a procedure similar to that outlined in Ref. 5. From Eq. (2.17) with $\xi(0,k)=d$ and $\eta(0,k)=0$ for all $k$, we have

$$\xi(t,K) = d \sum_{k=0}^{K} (-1)^k \exp(-k\Delta/T) R_1(t,k) \quad (3.25)$$

and

$$\xi(t,\infty) = d \sum_{k=0}^{\infty} (-1)^k \exp(-k\Delta/T) R_1(t,k) . \quad (3.26)$$

Subtracting Eq. (3.26) from Eq. (3.25) and substituting Eq. (2.18) for $R_1(t,k)$, we have

$$\xi(t,K) - \xi(t,\infty) = -\frac{d}{2\pi i} \oint du \cos\left[ \left[ \omega_0^2 + \frac{Gu^2}{1+u^2} \right]^{1/2} t \right] \sum_{k=K+1}^{\infty} \frac{(-1)^k}{u^{2k+1}} \exp(-k\Delta/T) . \quad (3.27)$$

Because

$$\sum_{k=K+1}^{\infty} \frac{(-1)^k}{u^{2k+1}} \exp(-k\Delta/T) = -\left[ \frac{(-1)^K}{u^{2K+1}} \exp(-K\Delta/T) \right] P(u) , \quad (3.28)$$

where

$$P(u)=\frac{e^{-\Delta/T}}{u^2+e^{-\Delta/T}} . \tag{3.29}$$

Equation (3.27) reduces to

$$\xi(t,K)-\xi(t,\infty)=d(-1)^K\exp(-K\Delta/T)\frac{1}{2\pi i}\oint du\frac{1}{u^{2K+1}}\cos\left[\left[\omega_0^2+\frac{Gu^2}{1+u^2}\right]^{1/2}t\right]P(u) . \tag{3.30}$$

We evaluate this integral using the saddle-point method. Comparing Eq. (3.30) with Eq. (3.1), we see that they differ only by the factor $P(u)$. Therefore, as a first approximation, we take the saddle points for Eq. (3.30) to be the same as those for Eq. (3.1). From Eqs. (B9) and (B4), we obtain the saddle points $u_s$ to be

$$u_s^2=e^{2i\theta_s}\cong-1\mp i\left[\frac{Gt}{2\omega_0 K}\right]^{1/2}e^{\pm i\pi/4} . \tag{3.31}$$

Carrying out an analysis similar to the one given in Appendix B, we get

$$\xi(t,K)-\xi(t,\infty)\cong\frac{d}{K\sqrt{8\pi}}\left[\frac{2KGt}{\omega_0}\right]^{1/4}\exp\left[-\frac{K\Delta}{T}+\left[\frac{KGt}{\omega_0}\right]^{1/2}\right]\mathrm{Re}\left\{P\exp\left[i\omega_0 t-i\frac{\pi}{8}+i\frac{Gt}{4\omega_0}-i\left[\frac{KGt}{\omega_0}\right]^{1/2}\right]\right\} , \tag{3.32}$$

where $P=P(u_s)$.

Because we are interested only in the amplitude of the oscillations, the absolute value of the above expression can be taken to give

$$|\xi(t,K)-\xi(t,\infty)|\cong\frac{d|P|}{K\sqrt{8\pi}}\left[\frac{2KGt}{\omega_0}\right]^{1/4}\exp\left[-\frac{K\Delta}{T}+\left[\frac{KGt}{\omega_0}\right]^{1/2}\right] , \tag{3.33}$$

where

$$|P|\cong\left[\frac{\Delta^2}{T^2}+\left[\frac{Gt}{2\omega_0 K}\right]-\left[\frac{Gt}{\omega_0 K}\right]^{1/2}\frac{\Delta}{T}\right]^{-1/2} . \tag{3.34}$$

The maximum transient amplitude is then obtained as

$$|\xi_m(t,K_m)-\xi(t,\infty)|\cong\frac{d|P_m|}{K_m\sqrt{8\pi}}\left[\frac{2K_m Gt}{\omega_0}\right]^{1/4}\exp\left[-\frac{K_m\Delta}{T}+\left[\frac{K_m Gt}{\omega_0}\right]^{1/2}\right] , \tag{3.35}$$

where

$$|P_m|\cong\left[\frac{\Delta^2}{T^2}+\left[\frac{Gt}{2\omega_0 K_m}\right]-\left[\frac{Gt}{\omega_0 K_m}\right]^{1/2}\frac{\Delta}{T}\right]^{-1/2} . \tag{3.36}$$

The results using Eq. (3.35) agree with numerical simulations showing that the FEL's listed in Table I exhibit no serious transient excursions. The results using Eq. (3.35) are also compared with numerical simulations for our hypothetical FEL in Table II. Good agreement between the two results is seen.

TABLE II. Comparison of numerical and theoretical values for maximum transient amplitude when bunches come in with equal initial displacement.

| $\omega_0$ ($10^9$ s$^{-1}$) | $\lvert\xi_m(t,K_m)-\xi(t,\infty)\rvert$ (cm) Simulation | Theory |
|---|---|---|
| 3.0 | 0.2255 | 0.2186 |
| 2.5 | 0.3131 | 0.3121 |
| 2.0 | 0.5503 | 0.5470 |
| 1.5 | 1.5973 | 1.4575 |

## IV. SUMMARY

The effects of a resistive wall on the transverse motion of a beam were studied. An expression for the transient-state solution was obtained for the situation where the focusing force dominates the resistive-wall force. This was accomplished using the integral representation of the complete solution. Expressions for the maximum transient-state amplitude were derived for three different cases: (a) single pulse, (b) random initial displacement, and (c) equal initial displacement. This led to the following criteria for the transverse resistive-wall instability to become significant: (a) $\Omega^2\geq\omega^2$, and (b) $\xi_m(t,K_m)\geq 1$ [cf. Eq. (3.4) or Eq. (3.35)] or $(\langle\xi^2(t,\infty)\rangle^{1/2})/(\langle\xi_0^2\rangle^{1/2}\geq 1)$ [cf. Eq. (3.24)]. Five proposed FEL's were studied using numerical simulations. Transverse resistive-wall instability did not affect the operation of any one of them.

**ACKNOWLEDGMENTS**

This work was supported by the Advanced Projects Agency of the Office of Basic Energy Sciences of the U.S. Department of Energy. The authors would like to thank Dr. R. L. Gluckstern, Dr. F. Neri, Dr. P. J. Channell, Dr. V. K. Neil, and Dr. R. K. Cooper for useful discussions. G. Rangarajan would also like to thank Dr. A. Dragt for his encouragement and support and Group AT-6, Los Alamos National Laboratory, for the warm hospitality he received.

## APPENDIX A: DERIVATION OF AN INTEGRAL REPRESENTATION OF $\xi(t,K)$

This appendix describes the derivation of an integral representation of $\xi(t,K)$. From Eq. (2.16)

$$\xi(t,K)=\sum_{k=0}^{K}(-1)^k\exp(-k\Delta/T)[R_1(t,k)\xi(0,K-k)+R_2(t,k)\eta(0,K-k)] \tag{A1}$$

where

$$R_1(t,k)=\sum_{n=0}^{\infty}\left[\frac{\pi\omega_0 t}{2}\right]^{1/2}\left[\frac{Gt}{2\omega_0}\right]^n\frac{1}{n!}\begin{bmatrix}k-1\\k-n\end{bmatrix}J_{n-1/2}(\omega_0 t)\xi(0,K-k) \tag{A2}$$

and

$$R_2(t,k)=\sum_{n=0}^{\infty}\left[\frac{\pi t}{2\omega_0}\right]^{1/2}\left[\frac{Gt}{2\omega_0}\right]^n\frac{1}{n!}\begin{bmatrix}k-1\\k-n\end{bmatrix}J_{n+1/2}(\omega_0 t)\eta(0,K-k)\ . \tag{A3}$$

It can be shown that

$$\begin{bmatrix}k-1\\k-n\end{bmatrix}=(-1)^{n-m}C^n_{2m-2n}(0)\ , \tag{A4}$$

where $C^n_{2m-2n}(0)$ is the Gegenbauer polynomial evaluated at zero. Its integral representation is given as

$$C^n_{2m-2n}(0)=\frac{1}{2\pi i}\oint\frac{du}{u^{2m-2n+1}}\frac{1}{(1+u^2)^n}\ . \tag{A5}$$

The contour in the complex $u$ plane encloses the origin and $|u|<1$ everywhere on the contour. Letting

$$\omega_0 t=x\ ,$$
$$G^{1/2}t=y\ , \tag{A6}$$

and using the following series representation of the Bessel function:

$$J_n(x)=\sum_{m=0}^{\infty}\frac{(-1)^m(x/2)^{2m+n}}{m!(m+n)!}\ , \tag{A7}$$

the expressions for $R_1(t,k)$ and $R_2(t,k)$ reduce to

$$R_1(t,k)=\frac{1}{2\pi i}\oint\frac{du}{u^{2k+1}}\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l)!}\sum_{m=0}^{\infty}\frac{l!(x^2)^m}{m!(l-m)!}\left[\frac{y^2u^2}{1+u^2}\right]^{l-m} \tag{A8}$$

and

$$R_2(t,k)=\frac{1}{2\pi i}\oint\frac{du}{u^{2k+1}}\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l+1)!}\sum_{m=0}^{\infty}\frac{tl!(x^2)^m}{m!(l-m)!}\left[\frac{y^2u^2}{1+u^2}\right]^{l-m}\ , \tag{A9}$$

where $l=n+m$. Noticing that

$$\sum_{m=0}^{\infty}\frac{l!(x^2)^m}{m!(l-m)!}\left[\frac{y^2u^2}{1+u^2}\right]^{l-m}=\left[x^2+\frac{y^2u^2}{1+u^2}\right]^l\ , \tag{A10}$$

Eqs. (A8) and (A9) can be written as

$$R_1(t,k)=\frac{1}{2\pi i}\oint\frac{du}{u^{2k+1}}\sum_{l=0}^{\infty}\frac{(-1)^l}{(2l)!}\left[x^2+\frac{y^2u^2}{1+u^2}\right]^l \tag{A11}$$

and

$$R_2(t,k)=\frac{1}{2\pi i}\oint\frac{du}{u^{2k+1}}\sum_{l=0}^{\infty}\frac{(-1)^l t}{(2l+1)!}\left[x^2+\frac{y^2u^2}{1+u^2}\right]^l . \tag{A12}$$

Using the standard power series expansions for cosine and sine functions and Eq. (A6), the final expressions for $R_1(t,k)$ and $R_2(t,k)$ are

$$R_1(t,k)=\frac{1}{2\pi i}\oint du\frac{1}{u^{2k+1}}\cos\left[\left[\omega_0^2+\frac{Gu^2}{1+u^2}\right]^{1/2}t\right] \tag{A13}$$

and

$$R_2(t,k)=\frac{1}{2\pi i}\oint du\left[\omega_0^2+\frac{Gu^2}{1+u^2}\right]^{-1/2}\frac{1}{u^{2k+1}}\sin\left[\left[\omega_0^2+\frac{Gu^2}{1+u^2}\right]^{1/2}t\right] . \tag{A14}$$

## APPENDIX B: A SADDLE-POINT EVALUATION OF $\xi(t,K)$ FOR A SINGLE PULSE

A saddle-point evaluation of

$$\xi(t,K)=d(-1)^K\exp(-K\Delta/T)\frac{1}{2\pi i}\oint du\frac{1}{u^{2k+1}}\cos\left[\left[\omega_0^2+\frac{Gu^2}{1+u^2}\right]^{1/2}t\right] \tag{B1}$$

is given in this Appendix.

Letting $u=e^{i\theta}$ in Eq. (B1), we get

$$\xi(t,K)=d(-1)^K\exp(-K\Delta/T)\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\, e^{-2iK\theta}\cos\left[\left[\omega_0^2+\frac{Ge^{i\theta}}{2\cos\theta}\right]^{1/2}t\right] . \tag{B2}$$

Because $|u|<1$, the imaginary part of $\theta$ has to be greater than or equal to zero. Assuming that $\omega_0^2 \gg G$, the square root can be expanded to give

$$\xi(t,K)=d(-1)^K\exp(-K\Delta/T)\frac{1}{4\pi}\int_{-\pi}^{\pi}d\theta\, e^{-2iK\theta}\left[\exp\left[i\omega_0 t+i\frac{\alpha}{2}-\alpha\frac{\tan\theta}{2}\right]+\exp\left[-i\omega_0 t-i\frac{\alpha}{2}+\alpha\frac{\tan\theta}{2}\right]\right] , \tag{B3}$$

where

$$\alpha=\frac{Gt}{2\omega_0} . \tag{B4}$$

Consider the first term [denoted by $\xi_I(t,K)$] in Eq. (B3). It can be rewritten as

$$\xi_I(t,K)=d(-1)^K\exp(-K\Delta/T)\frac{1}{4\pi}\int_{-\pi}^{\pi}d\theta\, e^{-2iK\theta}e^{tf(\theta)} , \tag{B5}$$

where

$$f(\theta)=-\frac{2iK\theta}{t}+i\omega_0+i\frac{\alpha}{2t}-\alpha\frac{\tan\theta}{2t} . \tag{B6}$$

The saddle points are given by

$$\frac{df}{d\theta}=0=-\frac{2iK}{t}-\alpha\frac{\sec^2\theta}{2t} \tag{B7}$$

and

$$\cos^2\theta=e^{i\pi/2}\frac{\alpha}{4K} . \tag{B8}$$

For parameters of interest, $\cos^2\theta \ll 1$. Hence, the two valid saddle points are given by

$$\theta_{1,2}\cong\pm\frac{\pi}{2}+e^{i\pi 4}\left[\frac{\alpha}{4K}\right]^{1/2} . \tag{B9}$$

Evaluating $f(\theta)$ at these saddle points, we find

$$tf(\theta_{1,2}) \cong \mp iK\pi + i\omega_0 t + i\frac{\alpha}{2} - i(2K\alpha)^{1/2} + (2K\alpha)^{1/2} , \tag{B10}$$

and $\xi_I(t,K)$ is therefore given by

$$\xi_I(t,K) \cong \frac{d}{K\sqrt{32\pi}} \left[ \frac{2KGt}{\omega_0} \right]^{1/4} \exp\left[ -\frac{K\Delta}{T} + \left[ \frac{KGt}{\omega_0} \right]^{1/2} \right] \exp\left[ i\omega_0 t - i\frac{\pi}{8} + i\frac{Gt}{4\omega_0} - i\left[ \frac{KGt}{\omega_0} \right]^{1/2} \right] . \tag{B11}$$

The second term in Eq. (B3) gives a contribution that is a complex conjugate to the one above. Adding the two contributions, the final result is

$$\xi(t,K) \cong \frac{d}{K\sqrt{8\pi}} \left[ \frac{2KGt}{\omega_0} \right]^{1/4} \exp\left[ -\frac{K\Delta}{T} + \left[ \frac{KGt}{\omega_0} \right]^{1/2} \right] \mathrm{Re}\left\{ \exp\left[ i\omega_0 t - i\frac{\pi}{8} + i\frac{Gt}{4\omega_0} - i\left[ \frac{KGt}{\omega_0} \right]^{1/2} \right] \right\} . \tag{B12}$$

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