General stability analysis of synchronized dynamics in coupled systems

Yonghong Chen,[1,3] Govindan Rangarajan,[2] and Mingzhou Ding[3]
[1]*Institute of Nonlinear Dynamics, Xi'an Jiaotong University, Xi'an 710049, Peoples Republic of China*
[2]*Department of Mathematics and Center for Theoretical Studies, Indian Institute of Science, Bangalore 560012, India*
[3]*Center for Complex Systems and Brain Sciences, Florida Atlantic University, Boca Raton, Florida 33431*
(Received 27 August 2002; revised manuscript received 25 November 2002; published 18 February 2003)

We consider the stability of synchronized states (including equilibrium point, periodic orbit, or chaotic attractor) in arbitrarily coupled dynamical systems (maps or ordinary differential equations). We develop a general approach, based on the master stability function and Gershgörin disk theory, to yield constraints on the coupling strengths to ensure the stability of synchronized dynamics. Systems with specific coupling schemes are used as examples to illustrate our general method.

DOI: 10.1103/PhysRevE.67.026209 PACS number(s): 05.45.Xt, 84.30.Ng

Large networks of coupled dynamical systems that exhibit synchronized static, periodic, or chaotic dynamics are subjects of great interest in a variety of fields ranging from biology [1] to semiconductor lasers [2] to electronic circuits [3]. For a given problem it is essential to know the extent to which the coupling strengths can be varied so that the synchronized state remains stable. Early attempts [4,5] at this question have typically looked either at systems of very small size or at very specific coupling schemes (diffusive coupling, global all to all coupling, etc., with a single coupling strength). Recent work [6,7] introduced the notion of a master stability function that enables the analysis of general coupling topologies. This function defines a region of stability in terms of the eigenvalues of the coupling matrix. In this paper we present a general method that provides explicit constraints on the coupling strengths themselves by combining the master stability function with the Gershgörin disk theory. Our approach is applicable to both coupled maps and coupled ordinary differential equations (ODEs). Commonly studied coupling schemes are used as illustrative examples.

Coupled maps. The system we consider is represented by

$$\mathbf{x}^i(n+1)=\mathbf{f}(\mathbf{x}^i(n))+\frac{1}{N}\sum_{j=1}^{N} G_{ij}\mathbf{H}(\mathbf{x}^j(n)), \tag{1}$$

where $\mathbf{x}^i(n)$ is the M-dimensional state vector of the ith map at time n and $\mathbf{H}:R^M\to R^M$ is the coupling function. We define $\mathbf{G}=[G_{ij}]$ as the coupling matrix, where G_{ij} gives the coupling strength from map j to map i. The condition $\Sigma_j G_{ij}=0$ is imposed to ensure that synchronized dynamics is a solution to Eq. (1).

Linearizing Eq. (1) around the synchronized state $\mathbf{x}(n)$, which evolves according to $\mathbf{x}(n+1)=\mathbf{f}(\mathbf{x}(n))$, we have

$$\mathbf{z}^i(n+1)=\mathbf{J}(\mathbf{x}(n))\cdot\mathbf{z}^i(n)+\frac{1}{N}\sum_{j=1}^{N} G_{ij}D\mathbf{H}(\mathbf{x}(n))\cdot\mathbf{z}^j(n), \tag{2}$$

where $\mathbf{z}^i(n)$ denotes the ith map's deviations from $\mathbf{x}(n)$, $\mathbf{J}(\cdot)$ is the $M\times M$ Jacobian matrix for $\mathbf{f}$, and $D\mathbf{H}(\cdot)$ is the Jacobian of the coupling function $\mathbf{H}$. In terms of the $M\times N$ matrix $\mathbf{S}(n)=(\mathbf{z}^1(n)\ \mathbf{z}^2(n)\ \cdots\ \mathbf{z}^N(n))$, Eq. (2) can be recast as

$$\mathbf{S}(n+1)=\mathbf{J}(\mathbf{x}(n))\cdot\mathbf{S}(n)+\frac{1}{N}D\mathbf{H}(\mathbf{x}(n))\cdot\mathbf{S}(n)\cdot\mathbf{G}^T. \tag{3}$$

According to the theory of Jordan canonical forms, the stability of Eq. (3) is determined by the eigenvalue λ of $\mathbf{G}$. Denote the corresponding eigenvector by $\mathbf{e}$ and let $\mathbf{u}(n)=\mathbf{S}(n)\mathbf{e}$. Then

$$\mathbf{u}(n+1)=\left(\mathbf{J}(\mathbf{x}(n))+\frac{1}{N}\lambda D\mathbf{H}(\mathbf{x}(n))\right)\cdot\mathbf{u}(n). \tag{4}$$

So the stability problem originally formulated in the $M\times N$ space has been reduced to a problem in an $M\times M$ space where it is often the case that $M\ll N$. It is worth mentioning that this eigenvalue based analysis is valid even if the coupling matrix $\mathbf{G}$ is defective [8].

We note that $\lambda=0$ is always an eigenvalue of $\mathbf{G}$ and its corresponding eigenvector is $(1\ 1\cdots 1)^T$ due to the synchronization constraint $\Sigma_{j=1}^N G_{ij}=0$. In this case, Eq. (4) can be used to generate the Lyapunov exponents for the individual system, which we denote by $h_1=h_{max}\geqslant h_2\geqslant\cdots\geqslant h_M$. These exponents describe the dynamics within the synchronization manifold defined by $\mathbf{x}^i=\mathbf{x}\ \forall\ i$.

The subspace spanned by the remaining eigenvectors is transverse to the synchronization manifold, in which the dynamics will be stable if the transverse Lyapunov exponents are all negative [9]. To examine this problem, we treat λ in Eq. (4) as a complex parameter and calculate the maximum Lyapunov exponent μ_{max} as a function of λ. This function is referred to as the master stability function by Pecora and Carroll [6]. The region in the $[\mathrm{Re}(\lambda),\mathrm{Im}(\lambda)]$ plane where $\mu_{max}<0$ defines a stability zone denoted by Ω. Figure 1 shows a schematic of two possible configurations of Ω. Whether Ω is an unbounded area [Fig. 1(a)] or a bounded one [Fig. 1(b)] is contingent on the coupling scheme and other system parameters. The origin, which is the zero eigenvalue of $\mathbf{G}$, may or may not lie in the stability zone. For example, for equilibrium or periodic state in coupled maps, the origin is in Ω, but for chaos, it lies outside of Ω. We note that, typically, Ω is obtained numerically. In some instances analytical results are possible (see below).

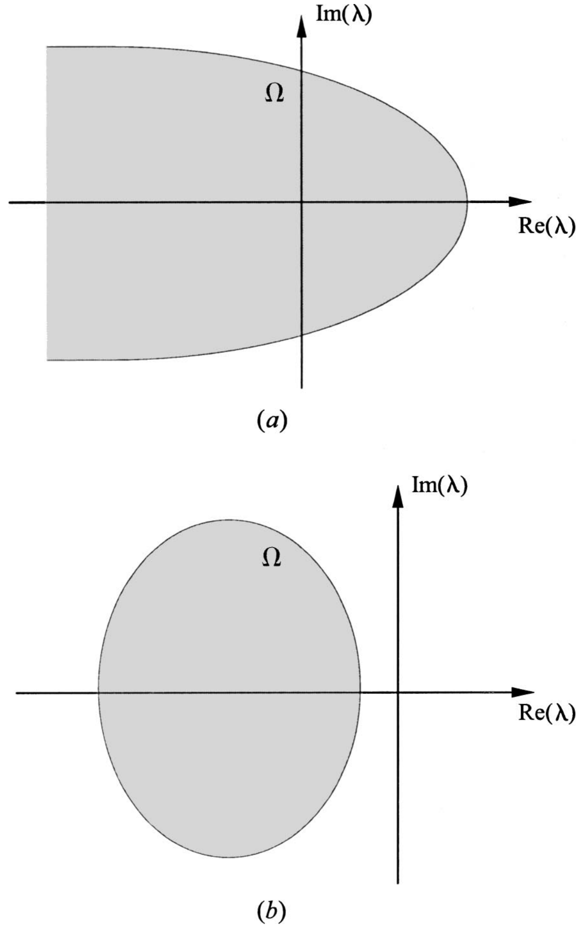

FIG. 1. Schematic illustrations of the stability zone: (a) unbounded area, (b) bounded area.

Clearly, if all the transverse eigenvalues of $\mathbf{G}$ lie within Ω, then the synchronized state is stable. Here we seek constraints applicable directly to the coupling strengths. This problem is dealt with by combining the master stability function with the Gershgörin disk theory.

The Gershgörin disk theorem [10] states that all the eigenvalues of an $n\times n$ matrix $\mathbf{A}=[a_{ij}]$ are located in the union of n disks (called Gershgörin disks), where each disk is given by

$$\left\{ z\in\mathbb{C}:|z-a_{ii}|<\sum_{j\neq i}|a_{ji}|\right\}, i=1,2,\ldots,n. \tag{5}$$

To apply this theorem to the transverse eigenvalues, we need to remove $\lambda=0$. We apply an order reduction technique in matrix theory [11], which leads to a $(N-1)\times(N-1)$ matrix $\mathbf{D}$ whose eigenvalues are the same as the eigenvalues of $\mathbf{G}$ except for $\lambda=0$.

Suppose that, for a given matrix $\mathbf{G}$, we have the knowledge of one of its eigenvalues $\tilde{\lambda}$ and the eigenvector $\mathbf{e}$. Through proper normalization we can make any component of $\mathbf{e}$ equal to 1. Here, without loss of generality, we assume that the first component is made equal to 1, namely, $\mathbf{e}=(1,\mathbf{e}_{N-1}^T)^T$. Rewrite $\mathbf{G}$ in the following block form:

$$\mathbf{G}=\begin{pmatrix} G_{11} & \mathbf{r}^T \\ \mathbf{s} & \mathbf{G}_{N-1}\end{pmatrix} \tag{6}$$

with $\mathbf{r}=(G_{12},\ldots,G_{1N})^T$, $\mathbf{s}=(G_{21},\ldots,G_{N1})^T$, and

$$\mathbf{G}_{N-1}=\begin{pmatrix} G_{22} & \cdots & G_{2N} \\ \vdots & \vdots & \vdots \\ G_{N2} & \cdots & G_{NN}\end{pmatrix}. \tag{7}$$

Choose a matrix $\mathbf{P}$ in the form

$$\mathbf{P}=\begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{e}_{N-1} & \mathbf{I}_{N-1}\end{pmatrix}. \tag{8}$$

Here $\mathbf{I}_{N-1}$ is the $(N-1)\times(N-1)$ identity matrix. Similarity transformation of $\mathbf{G}$ by $\mathbf{P}$ yields

$$\mathbf{P}^{-1}\cdot\mathbf{G}\cdot\mathbf{P}=\begin{pmatrix} \tilde{\lambda} & \mathbf{r}^T \\ \mathbf{0} & \mathbf{G}_{N-1}-\mathbf{e}_{N-1}\mathbf{r}^T\end{pmatrix}. \tag{9}$$

Since $\mathbf{P}^{-1}\mathbf{G}\mathbf{P}$ and $\mathbf{G}$ have identical eigenvalue spectra, the $(N-1)\times(N-1)$ matrix

$$\mathbf{D}^1=\mathbf{G}_{N-1}-\mathbf{e}_{N-1}\mathbf{r}^T \tag{10}$$

assumes the eigenvalues of $\mathbf{G}$ sans $\tilde{\lambda}$. We can obtain N different versions of the reduced matrix, which we denote by $\mathbf{D}^k$ $(k=1,2,\ldots,N)$, depending on which component of $\mathbf{e}$ is made equal to 1.

Applying the above technique to the coupling matrix $\mathbf{G}$ by letting $\tilde{\lambda}=0$ and $\mathbf{e}=(1\ 1\ \ldots\ 1)^T$ we get $\mathbf{D}^k=[d_{ij}^k]$, where $d_{ij}^k=G_{ij}-G_{kj}$. From the Gershgörin theorem the stability conditions of the synchronized dynamics are expressed as follows:

(1) The center of every Gershgörin disk of $\mathbf{D}^k$ lies inside the stability zone Ω. That is, $(G_{ii}-G_{ki},0)\in\Omega$.

(2) The radius of every Gershgörin disk of $\mathbf{D}^k$ satisfies the inequality

$$\sum_{j=1,j\neq i}^{N}|G_{ji}-G_{ki}|<\delta(G_{ii}-G_{ki}), i=1,2,\ldots,N \text{ and } i\neq k.$$

Here $\delta(x)$ is the distance from point x on the real axis to the boundary of the stability zone Ω.

As k varies from 1 to N, we obtain N sets of stability conditions. Each one provides sufficient conditions constraining the coupling strengths.

Coupled ODEs. The above procedure for obtaining stability bounds can also be applied to coupled identical ODEs written as

$$\dot{\mathbf{x}}^i=\mathbf{F}(\mathbf{x}^i)+\frac{1}{N}\sum_{j=1}^{N}G_{ij}\mathbf{H}(\mathbf{x}^j), \tag{11}$$

where $\mathbf{x}^i$ is the M-dimensional vector of the ith node. The dynamics of the individual node is $\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x})$. Linearizing around the synchronized state, we get

$$\dot{\mathbf{z}}^i = \mathbf{J}(\mathbf{x}) \cdot \mathbf{z}^i + \frac{1}{N} \sum_{j=1}^{N} G_{ij} D\mathbf{H}(\mathbf{x}) \cdot \mathbf{z}^j, \tag{12}$$

where $\mathbf{z}^i$ denotes deviations from $\mathbf{x}$, $\mathbf{J}(\cdot)$ and $D\mathbf{H}(\cdot)$ are the $M\times M$ Jacobian matrices for the functions $\mathbf{F}$ and $\mathbf{H}$, respectively. Adopting the Jordan canonical form, we obtain

$$\dot{\mathbf{u}} = \left[\mathbf{J}(\mathbf{x}) + \frac{1}{N} \lambda D\mathbf{H}(\mathbf{x}) \right] \mathbf{u}, \tag{13}$$

where λ is an eigenvalue of $\mathbf{G}$. Performing the same analysis as for coupled maps, we obtain the same stability conditions as given above.

Examples. We now illustrate the general approach by applying the above results to two examples where analytical results are possible. In the first example we consider the coupled differential equation systems with $\mathbf{H}(\mathbf{x}) = \mathbf{x}$ [7]. It is easy to see that $D\mathbf{H}$ is an $M\times M$ identity matrix. The Lyapunov exponents for Eq. (13) are easily calculated since the identity matrix commutes with $\mathbf{J}(\mathbf{x})$. Denoting them by $\mu_1(\lambda),\ \mu_2(\lambda),\ \dots,\ \mu_M(\lambda)$, we have

$$\mu_i(\lambda) = h_i + \frac{1}{N}\mathrm{Re}(\lambda), i = 1,2, \dots, M. \tag{14}$$

For stability, we require the transverse Lyapunov exponents $(\lambda \neq 0)$ to be negative. This is equivalent to the statement that the maximum Lyapunov exponent is less than zero:

$$\mu_{max}(\lambda) = h_{max} + \frac{1}{N}\mathrm{Re}(\lambda) < 0. \tag{15}$$

In other words, the stability zone Ω is the region defined by $\mathrm{Re}(\lambda) < -Nh_{max}$. The distance function from the center of each Gershgörin disk to the stability boundary is given by $\delta(G_{ii} - G_{ki}) = -Nh_{max} - (G_{ii} - G_{ki}) \quad (i = 1, \dots, N, i \neq k)$. Thus, the kth set of stability conditions is

$$(G_{ii} - G_{ki}) < -Nh_{max}, \tag{16}$$

$$\sum_{j=1, j\neq i}^{N} |G_{ji} - G_{ki}| < -Nh_{max} - (G_{ii} - G_{ki}),$$

$$i = 1,2, \dots, N, \quad i \neq k. \tag{17}$$

It is obvious that the second inequality implies the first one. So the stability condition for the synchronized state (whether an equilibrium, periodic, or chaotic state) is given by

$$\sum_{j=1, j\neq i}^{N} |G_{ji} - G_{ki}| + (G_{ii} - G_{ki}) < -Nh_{max},$$

$$i = 1,2, \dots, N, \quad i \neq k. \tag{18}$$

When the coupling is symmetric, i.e., $G_{ij} = G_{ji}$, Rangarajan and Ding [12], based on the use of Hermitian and positive semidefinite matrices, derived a very simple stability constraint

$$G_{ij} > h_{max}, \quad \forall \quad i,j. \tag{19}$$

We show here that Eq. (19) is a consequence of the more general stability conditions given in Eq. (18). This can be seen as follows. First consider $k = 1$. Substituting $G_{ii} = -\sum_{j=1, j\neq i}^{N} G_{ji}$ (synchronization condition) and simplifying we get the following equation:

$$\sum_{j=2, j\neq i}^{N} |G_{ji} - G_{1i}| - \sum_{j=2, j\neq i}^{N} G_{ji} - 2G_{1i} < -Nh_{max}, i \neq 1. \tag{20}$$

If $G_{ji} - G_{1i}$ is positive for all allowed i and j values, it is easy to see that the above stability condition is satisfied given the condition in Eq. (19). However, if more than two such terms are negative, we have a problem. We can get around this by considering the other $(N-1)$ sets of stability conditions obtained by setting $k = 2,3, \dots, N$ in Eq. (18):

$$\sum_{j=1, j\neq i\neq 2}^{N} |G_{ji} - G_{2i}| - \sum_{j=1, j\neq i\neq 2}^{N} G_{ji} - 2G_{2i} < -Nh_{max},$$

$$i \neq 2$$

$$\vdots \tag{21}$$

$$\sum_{j=1, j\neq i}^{N-1} |G_{ji} - G_{Ni}| - \sum_{j=1, j\neq i}^{N-1} G_{ji} - 2G_{Ni} < -Nh_{max}, \quad i \neq N.$$

If we take the average of the inequalities over k, cancellation takes place, resulting in a simplified inequality that will be satisfied if the sufficient condition given in Eq. (19) is met. In other words, the previously derived stability condition is obtained as a special case when we require the coupling strengths to meet the N stability conditions simultaneously.

In the second example, we consider a coupled map with $\mathbf{H} = \mathbf{f}$ [5]. Under this assumption, $D\mathbf{H} = \mathbf{J}$ and the linearized equation [cf. Eq. (4)] reduces to

$$\mathbf{u}(n+1) = (\lambda/N + 1)\mathbf{J}(\mathbf{x}(n))\mathbf{u}(n). \tag{22}$$

The Lyapunov exponents for Eq. (22) are easily calculated analytically. Denoting them by $\mu_1(\lambda),\ \mu_2(\lambda),\ \dots,\ \mu_M(\lambda)$, we have

$$\mu_i(\lambda) = h_i + \ln|\lambda/N + 1|,\ i = 1,2, \dots, M. \tag{23}$$

For stability, we require $\mu_{max}(\lambda) = h_{max} + \ln|\lambda/N + 1| < 0$. In other words, the stability zone is defined by

$$|\lambda + N| < N\exp(-h_{max}). \tag{24}$$

The distance from the center of each Gershgörin disk to the boundary is easily calculated to be $\delta(G_{ii} - G_{ki}) = N\exp(-h_{max}) - |N + G_{ii} - G_{ki}|$ $(i = 1, \dots, N,\ i \neq k)$. Thus, the conditions of stability are

$$\sum_{j=1,j\neq i}^{N} |G_{ji}-G_{ki}|+|N+(G_{ii}-G_{ki})|<N\exp(-h_{max}),$$

$$i=1,\ldots,N,\ i\neq k,\ k=1 \text{ or } 2 \text{ or } \cdots \text{ or } N. \quad (25)$$

For each k from 1 to N, we obtain a set of sufficient stability conditions.

In Ref. [12], a simple stability bound for synchronized chaos in the case of symmetric coupling was obtained as

$$[1-\exp(-h_{max})]<G_{ij}<[1+\exp(-h_{max})], \quad \forall \quad i,j. \quad (26)$$

This can again be derived from the general stability condition in Eq. (25) with the averaging technique used above.

In summary, we have set up a general formalism to study the stability of synchronized state in coupled identical maps and ordinary differential equations. We have also considered the often used coupling function for coupled maps and coupled ODEs and given analytical results in these cases. We have also shown that known stability bounds can be derived from our more general results.

This work was supported by U.S. ONR Grant No. N00014-99-1-0062.

[1] R.R. Klevecz, J. Bolen, and O. Duran, Int. J. Bifurcation Chaos Appl. Sci. Eng. **2**, 941 (1992); D. Hansel and H. Sompolinsky, Phys. Rev. Lett. **68**, 718 (1992); D. Hansel, Int. J. Neural Syst. **7**, 403 (1996); F. Pasemann, Physica D **128**, 236 (1999).

[2] H.G. Winful and L. Rahman, Phys. Rev. Lett. **65**, 1575 (1990); R. Li and T. Erneux, Opt. Commun. **99**, 196 (1993); Phys. Rev. A **49**, 1301 (1994); R. Roy and K.S. Thornburg, Jr., Phys. Rev. Lett. **72**, 2009 (1994); K. Otsuka, R. Kawai, S. Hwong, J. Ko, and J. Chern, *ibid.* **84**, 3049 (2000).

[3] C.W. Wu and L.O. Chua, IEEE Trans. Circuits Syst., I: Fundam. Theory Appl. **42**, 430 (1995); S. Jankowski, A. Londei, C. Mazur, and A. Lozowski, Int. J. Electron. **79**, 823 (1995); G. Filatrella, B. Straughn, and P. Barbara, J. Appl. Phys. **90**, 5675 (2001).

[4] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **69**, 32 (1983); I. Waller and R. Kapral, Phys. Lett. **105A**, 163 (1984); V.S. Afraimovich *et al.*, Sov. Radiophys. **29**, 795 (1986); G.L. Oppo and R. Kapral, Phys. Rev. A **36**, 5820 (1987); L.A. Bunimovich and Ya.G. Sinai, Nonlinearity **1**, 491 (1988); J.F. Heagy, L.M. Pecora, and T.L. Carroll, Phys. Rev. E **50**, 1874 (1994); L. Kocarev and U. Parlitz, Phys. Rev. Lett. **76**, 1816 (1996); R. Brown and N.F. Rulkov, *ibid.* **78**, 4189 (1997); N.F. Rulkov and M.M. Sushchik, Int. J. Bifurcation Chaos Appl. Sci. Eng. **7**, 625 (1997); P.C. Bresloff, Phys. Rev. E **60**, 2160 (1999); P. Glendinning, Phys. Lett. A **259**, 129 (1999); V.N. Belykh, I.V. Belykh, and M. Hasler, Phys. Rev. E **62**, 6332 (2000); V.N. Belykh, I.V. Belykh, and E. Mosekilde, *ibid.* **63**, 036216 (2001).

[5] A.S. Pikovsky, Z. Phys. B: Condens. Matter **55**, 149 (1984); K. Kaneko, Prog. Theor. Phys. **72**, 480 (1984); K. Kaneko, Physica D **23**, 436 (1986); N. Chatterjee and N. Gupte, Phys. Rev. E **53**, 4457 (1996); D.J. Gauthier and J.C. Bienfang, Phys. Rev. Lett. **77**, 1751 (1996); M. Ding and W. Yang, Phys. Rev. E **56**, 4009 (1997); K. Zhu and T. Chen, *ibid.* **63**, 067201 (2001); J. Jost and M.P. Joy, *ibid.* **65**, 016201 (2002).

[6] L.M. Pecora and T.L. Carroll, Phys. Rev. Lett. **80**, 2109 (1998); K.S. Fink, G. Johnson, T.L. Carroll, and L.M. Pecora, Phys. Rev. E **61**, 5080 (2000); M. Barahona and L.M. Pecora, Phys. Rev. Lett. **89**, 054101 (2002).

[7] G. Hu, J. Yang, and W. Liu, Phys. Rev. E **58**, 4440 (1998); M. Zhan, G. Hu, and J. Yang, *ibid.* **62**, 2963 (2000).

[8] M. W. Hirsch and S. Smale, *Differential Equations, Dynamical Systems, and Linear Algebra* (Academic Press, New York, 1974).

[9] We note that since the Lyapunov exponent h_{max} is computed from typical initial conditions, the stability here is with respect to the blowout bifurcation. Bubbling transition can occur while the parameters are still within the bound derived in this work. For details on these phenomena see E. Ott and J.C. Sommerer, Phys. Lett. A **188**, 39 (1994); P. Ashwin, J. Buescu, and I. Stewart, Phys. Lett. **193**, 126 (1994); S.C. Venkataramani *et al.*, Phys. Rev. Lett. **77**, 5361 (1996); Phys. Rev. E **54**, 1346 (1996).

[10] R. A. Horn and C. R. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, 1985).

[11] G. H. Golub and C. F. Van Loan, *Matrix Computations* (Johns Hopkins University Press, Baltimore, 1996).

[12] G. Rangarajan and M. Ding, Phys. Lett. A **296**, 204 (2002).