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## Kinematic Moment Invariants for Linear Hamiltonian Systems

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Quadratic moments of a particle distribution being transported through a linear Hamiltonian system are considered. A complete set of kinematic invariants made out of these moments are constructed leading to the discovery of new invariants.

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Consider a distribution of particles being transported through a Hamiltonian system. An important first step towards a complete understanding of its evolution would be to determine quantities that remain invariant under this transport. Since a particle distribution is characterized by its moments, it is natural to seek invariant functions of these moments.

In this Letter, we construct quadratic functions of moments that are invariant under the action of a linear Hamiltonian system. These functions remain approximately invariant for Hamiltonian systems that are not strongly nonlinear. Hence they should be useful in a perturbative analysis of particle distribution evolution through such systems.

The moment invariants that we will construct are invariant for any linear Hamiltonian system and hence are called kinematic invariants. In contrast, dynamic invariants like constants of motion are invariant only for a given Hamiltonian. One quadratic moment invariant was already known.<sup>1</sup> We give a simple and practical method to construct a complete set of quadratic moment invariants. This method can easily be extended to construct higher-order moment invariants. However, quadratic moment invariants are the most important and hence we will restrict ourselves to these.

Let  $z = (q_1, p_1, q_2, p_2, q_3, p_3)$  be the six-dimensional vector describing the location of a particle in phase space. Consider the action of a linear Hamiltonian system on this particle. Because the system is linear, the final coordinates  $z^f$  are given as linear combinations of

initial coordinates  $z^i$ :

$$z_a^f = \sum_{b=1}^6 M_{ab} z_b^i. \quad (1)$$

This can be written compactly as

$$z^f = M z^i, \quad (2)$$

where  $M$  is a  $6 \times 6$  matrix. Since the particle's evolution is governed by a Hamiltonian, it can be shown<sup>2</sup> that  $M$  satisfies the symplectic condition

$$\tilde{M} J M = J, \quad (3)$$

where  $\tilde{M}$  is the transpose of  $M$  and

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \quad (4)$$

Matrices  $M$  satisfying Eq. (3) are called symplectic matrices and the set of all such  $6 \times 6$  matrices  $M$  forms the symplectic group  $\text{Sp}(6)$ .<sup>2</sup>

Now consider a (discrete) distribution of particles. Define the quadratic moments of this distribution to be the following:

$$\langle z_a z_b \rangle = \frac{1}{N} \sum_{i=1}^N (z_a^{(i)} z_b^{(i)}), \quad (5)$$

where  $N$  is the number of particles in the distribution. Under transport by  $M$  [cf. Eq. (2)], the moments transform as

$$\langle\langle z_a z_b \rangle\rangle^f = \sum_{c,d} \langle M_{ac} z_c M_{bd} z_d \rangle \quad (6)$$

or

$$\langle\langle z \bar{z} \rangle\rangle^f = M \langle\langle z \bar{z} \rangle\rangle \tilde{M}. \quad (7)$$

Next define the matrix  $E$ , with elements

$$E_{ab} = \langle z_a (\bar{z} J)_b \rangle. \quad (8)$$

Under time evolution,  $E$  transforms as [cf. Eqs. (2) and (3)]

$$E = \langle z (\bar{z} J) \rangle \rightarrow \langle M z (\bar{z} \tilde{M} J) \rangle = M \langle z (\bar{z} J) \rangle M^{-1},$$

or

$$E \rightarrow M E M^{-1}. \quad (9)$$

From Eq. (9), we see that the eigenvalues of  $E$  are invariant, since (9) is a similarity transformation. The matrix  $E$  was defined in terms of quadratic moments [cf. Eq. (8)] and hence its eigenvalues are also functions of quadratic moments. Since the eigenvalues were shown to be invariant, we have, in effect, succeeded in con-

structing functions of quadratic moment invariants that remain invariant for any linear Hamiltonian system.

We now explore the properties of these eigenvalues. We first restrict ourselves to the simple two-dimensional phase space, i.e.,  $z = (q_1, p_1)$ . In this case, the matrix  $E$  takes the form

$$E = \begin{pmatrix} -\langle q_1 p_1 \rangle & \langle q_1^2 \rangle \\ -\langle p_1^2 \rangle & \langle p_1 q_1 \rangle \end{pmatrix}. \quad (10)$$

One can easily compute the eigenvalues of  $E$  to be

$$\lambda = \pm i (\langle q_1^2 \rangle \langle p_1^2 \rangle - \langle q_1 p_1 \rangle^2)^{1/2} = \pm i \epsilon. \quad (11)$$

Using the "triangle inequality" among moments

$$|\langle q_1 p_1 \rangle|^2 \leq |\langle q_1^2 \rangle| |\langle p_1^2 \rangle|, \quad (12)$$

it is seen that  $\epsilon = (\langle q_1^2 \rangle \langle p_1^2 \rangle - \langle q_1 p_1 \rangle^2)^{1/2}$  is real. Hence  $\lambda$  is pure imaginary. Accelerator physicists will immediately recognize  $\epsilon$  to be nothing but the usual rms emittance<sup>3</sup> (up to numerical factors) which is well known to be an invariant.

Now consider the full six-dimensional phase space. The kinematic moment invariants can be constructed following the procedure given below. (1) Construct the matrix  $E$  using Eq. (9):

$$E = \begin{pmatrix} -\langle q_1 p_1 \rangle & \langle q_1^2 \rangle & -\langle q_1 p_2 \rangle & \langle q_1 q_2 \rangle & -\langle q_1 p_3 \rangle & \langle q_1 q_3 \rangle \\ -\langle p_1^2 \rangle & \langle q_1 p_1 \rangle & -\langle p_1 p_2 \rangle & \langle q_2 p_1 \rangle & -\langle p_1 p_3 \rangle & \langle q_3 p_1 \rangle \\ -\langle q_2 p_1 \rangle & \langle q_1 q_2 \rangle & -\langle q_2 p_2 \rangle & \langle q_2^2 \rangle & -\langle q_2 p_3 \rangle & \langle q_2 q_3 \rangle \\ -\langle p_1 p_2 \rangle & \langle q_1 p_2 \rangle & -\langle p_2^2 \rangle & \langle q_2 p_2 \rangle & -\langle p_2 p_3 \rangle & \langle q_3 p_2 \rangle \\ -\langle q_3 p_1 \rangle & \langle q_1 q_3 \rangle & -\langle q_3 p_2 \rangle & \langle q_2 q_3 \rangle & -\langle q_3 p_3 \rangle & \langle q_3^2 \rangle \\ -\langle p_1 p_3 \rangle & \langle q_1 p_3 \rangle & -\langle p_2 p_3 \rangle & \langle q_2 p_3 \rangle & -\langle p_3^2 \rangle & \langle q_3 p_3 \rangle \end{pmatrix}. \quad (13)$$

(2) Compute its six eigenvalues. They should be pure imaginary and come in complex-conjugate pairs (for a rigorous proof of this fact see Ref. 4). Denote the eigenvalues by  $\pm i\epsilon_1$ ,  $\pm i\epsilon_2$ , and  $\pm i\epsilon_3$ . (3) The quantities  $\epsilon_1$ ,  $\epsilon_2$ , and  $\epsilon_3$  are conserved under linear Hamiltonian transport. Since they are functions of quadratic moments, they form a complete set of kinematic moment invariants.

In accelerator-physics language, the  $\epsilon_i$ 's can be thought of as three independent eigenemittances. They are generalizations of the conventional emittances constructed in the  $q_1$ - $p_1$ ,  $q_2$ - $p_2$ , and  $q_3$ - $p_3$  planes. These conventional emittances are conserved only in the absence of coupling between the 3 degrees of freedom. The eigenemittances that we have constructed are conserved even when coupling is present.

If the coupling between different planes vanishes, these eigenemittances reduce to the conventional emittances as expected. This can be seen as follows. As the coupling

goes to zero, the cross terms in matrix  $E$  vanish, i.e.,

$$\langle q_i p_j \rangle = \langle q_i q_j \rangle = \langle p_i p_j \rangle = 0 \text{ for } i \neq j. \quad (14)$$

Thus, all the elements of the matrix  $E$  in Eq. (13) become zero, except for three 2-by-2 blocks along the diagonal. These blocks are identical in form to the matrix that we obtained in the one-dimensional case [cf. Eq. (10)]. Since eigenvalues of a block-diagonal matrix are given by the eigenvalues of its component blocks, the eigenvalues of  $E$  in the uncoupled case are found to be as follows [cf. Eq. (11)]:

$$\lambda_i = \pm i (\langle q_i^2 \rangle \langle p_i^2 \rangle - \langle q_i p_i \rangle^2)^{1/2} = \pm i \epsilon_i \quad (15)$$

for  $i = 1, 2, 3$ .

It is seen that  $\epsilon_i$  are nothing but the usual rms emittances in the three individual planes. Thus, when the coupling between the planes is zero, the three invariants

(eigenemittances) that we have constructed are identical to the conventional rms emittances for the three planes. Otherwise, they contain a mixture of the conventional emittances with cross terms present.

Finally, we note that an equivalent set of kinematic moment invariants can be constructed using the fact that  $\text{tr}(E^n)$  is also conserved under Eq. (9). Thus, we can take the invariants to be<sup>5</sup>

$$I_n = \text{tr}(E^n). \quad (16)$$

In particular,  $I_2$  is the quadratic moment invariant discovered by Lysenko<sup>1</sup> using a different method. The other two independent invariants are  $I_4$  and  $I_6$ . It can be shown that  $I_n$  for  $n$  odd is zero and that the other  $I_n$ 's are functionally dependent on  $I_2$ ,  $I_4$ , and  $I_6$ . This can be seen as follows. The matrix  $E$  is such that if  $\lambda$  is an eigenvalue, then  $-\lambda$  is also an eigenvalue. The six eigenvalues of  $E$  can be written as  $\pm\lambda_1$ ,  $\pm\lambda_2$ , and  $\pm\lambda_3$ . Therefore, from Eq. (16)

$$I_n = \lambda_1^n + (-\lambda_1)^n + \lambda_2^n + (-\lambda_2)^n + \lambda_3^n + (-\lambda_3)^n. \quad (17)$$

Thus,  $I_n = 0$  for  $n$  odd. Moreover, the nonzero  $I_n$ 's depend only on three independent parameters. Hence, only three of the  $I_n$ 's are functionally independent. These three functionally independent invariants are conveniently taken to be  $I_2$ ,  $I_4$ , and  $I_6$ .

To summarize, in this Letter we have developed a systematic method to construct kinematic moment invariants for linear Hamiltonian systems. This led to the

discovery of new invariants. These invariants should be useful for a perturbative analysis of the evolution of particle distributions through more general Hamiltonian systems.

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<sup>1</sup>W. Lysenko and M. Overly, in *Proceedings of the Workshop on Linear Accelerator and Beam Optics Codes, San Diego, CA, 1988*, edited by C. R. Eiminher, AIP Conference Proceedings No. 177 (American Institute of Physics, New York, 1988), p. 323; see also W. Lysenko, Los Alamos National Laboratory Technical Report No. LA-UR-88-165, 1988 (unpublished).

<sup>2</sup>For a review of the properties of symplectic matrices, see A. J. Dragt, in *Physics of High Energy Particle Accelerators*, edited by R. A. Carrigan, F. R. Huson, and M. Month, AIP Conference Proceedings No. 87 (American Institute of Physics, New York, 1982), p. 147; see also A. J. Dragt *et al.*, *Annu. Rev. Nucl. Part. Sci.* **38**, 455 (1988).

<sup>3</sup>P. Lapostolle, *IEEE Trans. Nucl. Sci.* **18**, 1101 (1971); F. J. Sacherer, *IEEE Trans. Nucl. Sci.* **18**, 1105 (1971).

<sup>4</sup>F. Neri, University of Maryland report, 1988 (to be published).

<sup>5</sup>A similar expression for the invariants was discovered independently by D. Holm, W. Lysenko, and C. Scovel, Eidgenössische Technische Hochschule Zurich report, 1989 (to be published).