Mid-Term Examination
for the course
Introduction to the Calculus of Variations
MA 374

Instructor:

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Attempt all questions and provide as much as details as possible. Please write in a clear and legible manner and submit only your own work.

1. Consider the minimization problem

$$\min \left\{ I\left[u\right] := \int_0^1 \left[\frac{\kappa}{2} \left|\dot{u}\left(t\right)\right|^2 - \rho u\left(t\right) \right] t \ \mathrm{d}t : u\left(1\right) = 0 \right\}.$$

   (a) Write down the Euler-Lagrange equation and find all extremals.

   (b) Find all extremals u with $I\left[u\right] < \infty$.

   (c) Show that

$$\inf_{u \in C^1([0,1])} \left\{ \frac{\kappa}{2} \int_0^1 \left|\dot{u}\left(t\right)\right|^2 t \ \mathrm{d}t - \rho u\left(0\right) : u\left(1\right) = 0 \right\} = -\infty.$$

2. Find the Euler-Lagrange equation and other conditions satisfied by any smooth extremal for the following variational problem.

$$\inf_{u \in C^2([a,b])} \left\{ I\left(u\right) = \int_a^b f\left(t, u\left(t\right), \dot{u}\left(t\right), \ddot{u}\left(t\right)\right) \ \mathrm{d}t \right\} = m.$$

3. Use classical methods to solve the following two minimization problems, where $\kappa, \rho > 0$ are constants.

(a)

$$\min\left\{\int_{-1}^{1}\left[\frac{\kappa}{2}\left|\ddot{u}\left(t\right)\right|^{2}-\rho u\left(t\right)\right]\,\mathrm{d}t: u\left(-1\right)=u\left(1\right)=\dot{u}\left(-1\right)=\dot{u}\left(1\right)=0\right\},$$

(b)

$$\min\left\{\frac{\kappa}{2}\int_{-1}^{1}\left|\ddot{u}\left(t\right)\right|^{2}\,\mathrm{d}t-\rho u\left(0\right): u\left(-1\right)=u\left(1\right)=\dot{u}\left(-1\right)=\dot{u}\left(1\right)=0\right\}.$$

4. Let $(x_1,y_1),(x_2,y_2)\in\mathbb{R}^2$ with $x_1<x_2$ and let $L>0$ be a real number. Use classical methods to find the general form of the trajectory which solves the following minimization problem

$$\inf_{u\in X}\left\{I\left[u\right]=\int_{x_1}^{x_2}u\left(x\right)\sqrt{1+\left|u'\left(x\right)\right|^{2}}\,\mathrm{d}x\right\}=m,$$

where

$$X=\left\{u\in C^{1}\left(\left[x_1,x_2\right]\right): \begin{array}{c} u\left(x_1\right)=y_1, u\left(x_2\right)=y_2 \\ \text{and } \displaystyle\int_{x_1}^{x_2}\sqrt{1+\left|u'\right|^{2}}\,\mathrm{d}x=L \end{array}\right\}.$$

5. Let $x:[t_0,t_1]\to\mathbb{R}^3$ and consider the functional

$$I\left[x\right]:=\int_{t_0}^{t_1}x_3\left(t\right)\sqrt{\dot{x_1}^{2}\left(t\right)+\dot{x_2}^{2}\left(t\right)+\dot{x_3}^{2}\left(t\right)}\,\mathrm{d}t.$$

Find the extremals for I subject to the constraint

$$g\left(x\right):=x_1^2+x_2^2-1=0$$

and the boundary conditions

$$x\left(t_0\right)=x_0 \quad \text{and } x\left(t_1\right)=x_1.$$

6. Let $\Sigma\subset\mathbb{R}^3$ be a smooth surface, implicitly given by the equation

$$g\left(x\right)=0,$$

where $g:\mathbb{R}^3\to\mathbb{R}$ is a smooth function such that $\nabla g\neq 0$ on the set $g^{-1}\left(0\right)$. Use the Lagrange multiplier technique to find the equation for the geodesic $\gamma:[t_0,t_1]\to\Sigma$, passing through two points p_1 and p_2 on Σ. Using the result, show that

(a) The geodesics on a sphere are the great circles.

(b) The geodesics on a right circular cylinder are helices.

7. Let $u : [0, T] \to \mathbb{R}$ and consider the functional

$$I[u] := \int_0^T \left[\dot{u}^2(t) - u^2(t)\right] \, \mathrm{d}t$$

along with the boundary conditions

$$u(0) = 0 = u(T).$$

(a) Write down the Euler-Lagrange equations and find its general solution.

(b) Show that if $\pi < T < 2\pi$, then the only extremal is $u \equiv 0$.

(c) Calculate the second variation at $u \equiv 0$ and write down the associated Jacobi equation.

(d) Show that the second variation at $u \equiv 0$ can be strictly negative.

(e) Show that there is a interior conjugate point in $[0, T]$ and find the conjugate point by exhibiting directly a nontrivial solution of the Jacobi equation vanishing at that point.