Introduction to the Calculus of Variations

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1 Course contents and outline

Our goal is to provide a gentle introduction to the direct methods in Calculus of Variations. We would restrict ourselves to methods concerning minizations problems. Methods to find other critical points which are not a local extrema (e.g. Min-Max methods, Mountain Pass techniques) would be excluded. The focus will be on illustrating the main methods using important prototype examples and not on proving the most general or the sharpest results. We would tentatively cover the following topics.

- **Classical Methods**

  Euler-Lagrange equations, Lagrangian and Hamiltonian formulations, Constrained problems and Lagrange multipliers, an illustration of the classical methods: Geodesic curves.

- **Direct Methods**

  Sobolev spaces, Poincaré and Sobolev inequalities, Dirichlet integral and p-Dirichlet Integral, Existence theorem for minimizers, examples and counterexamples, weak form of the Euler-Lagrange equations, Dirichlet Principle, discussion of quasiconvexity as a necessary condition for lower semicontinuity, polyconvexity and weak continuity of determinants.

- **Regularity:** Caccioppoli inequality, L^2 regularity.

- **Area functional**

  Plateau's problem and minimal surfaces, Parametric Plateau's problem: Douglas-Courant-Tonelli method, Nonparametric minimal surfaces.

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2   Main references

The basic references for this course are [3] and [5]. Advanced references which discuss most of our topics are [2] and [7]. An excellent references for minimal surfaces and CMC surfaces is [6]. For regularity questions, both [1] and [4] are excellent, albeit advanced references.

References

[1] AMBROSIO, L., CARLOTTO, A., AND MASSACCESI, A. *Lectures on elliptic partial differential equations*, vol. 18 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, 2018.

[2] DACOROGNA, B. *Direct methods in the calculus of variations*, second ed., vol. 78 of *Applied Mathematical Sciences*. Springer, New York, 2008.

[3] DACOROGNA, B. *Introduction to the calculus of variations*, third ed. Imperial College Press, London, 2015.

[4] GIAQUINTA, M., AND MARTINAZZI, L. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, second ed., vol. 11 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, 2012.

[5] JOST, J., AND LI-JOST, X. *Calculus of variations*, vol. 64 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1998.

[6] STRUWE, M. *Plateau's problem and the calculus of variations*, vol. 35 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1988.

[7] STRUWE, M. *Variational methods*, fourth ed., vol. 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2008. Applications to nonlinear partial differential equations and Hamiltonian systems.