Introduction to Partial Differential Equations

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1 Course Objective

To provide a thorough introduction to the classical theoretical/analytical methods in Partial Differential Equations. If time permits, we would discuss the modern methods for elliptic boundary value problems very briefly.

2 Prerequisites

- **Analysis** A course in Real analysis and a course in Measure and Integration is crucial. These would be used quite freely. L^p spaces would be used constantly.
- **ODE** The Cauchy-Peano-Picard-Lindelöf-Lipschitz theory of existence of solutions to Ordinary Differential Equations and solutions of simple ODEs would be used in the course.

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3 Course contents and outline

Our goal is to cover the following topics.

- **Part 1: Introductory material**
  - **Chapter 1: Introduction** Multiindex notation, Examples of different types of PDEs, linear and nonlinear equations, equations and systems. Hadamard's well-posedness notion. Different types of problems for PDEs, Cauchy problem or IVP, BVP, mixed problems. Brief idea of weak solutions.
  - **Chapter 2: Separation of Variables method**
    - Fourier series: definition, properties, half-range series, Bessel's inequality, Decay of Fourier coefficients, Dirichlet theorem and convergence of Fourier series, term by term differentiation and integration of Fourier series.
    - Separation of variables method for the heat equation, formal solution, convergence and C^∞ regularity.
    - Separation of variables method for one dimensional wave equation, formal solution, convergence and C^2 regularity of solutions.
    - Separation of variables method for Laplace equation in a rectangle and in a disk, formal solution, convergence and C^∞ regularity.
  - **Chapter 3: Basics of Linear differential operators**
    - Fourier transform in L^1 and L^2. Basic properties, Differentiation to multiplication, convolutions to products, Fourier inversion formula.
    - Basic behavior of linear PDEs: Symbol of a PDE, Characteristic variety, Cauchy problem and noncharacteristic hypersurfaces, Ellipticity and ill-posedness of the Cauchy problem, Cauchy problem and characteristic curves, Hyperbolicity.
    - Idea of a Fundamental solution: Mention of the Ehrenpreis-Malgrange theorem (No proof).
    - Hypoellipticity and analytic-hypoellipticity, parametrix.
- **Part 2: PDEs as propagation: Hyperbolic techniques**
  - **Chapter 4: The wave equation**
    - Constant coefficient linear Transport equation
    - Wave equation in one space dimension, D'Alembert's formula
    - Wave equation in higher dimensions
      - Spherical means, Euler-Poisson-Darboux equation,
      - Solution for $n = 3$, Kirchhoff formula.
      - Hadamard method of descent, solution for $n = 2$ and Poisson formula.

        - Solution for odd n and even n.
        - Energy methods, Domain of dependence, finite speed of propagation, Huygen's principle.
      - Some examples of plane and standing wave solutions and solitons
  - **Chapter 5: First order Hyperbolic equations**
      - Contact element and envelopes.
      - Method of characteristics, Characteristic strips, general characteristic ODEs, examples of solution by method of characteristic for linear and quasilinear equations
      - Fully nonlinear equations, Hamilton-Jacobi equations, the eikonal equation and Huygen's principle.
      - Shocks and rarefaction in nonlinear conservation laws, weak solutions, Rankine-Hugoniot jump conditions.
      - Power series and local existence with analytic coefficients: Cauchy-Kowalevski theory (basic idea), Holmgren uniqueness theorem (basic idea) and Lewy's example

- **Part 3: Elliptic and hypoelliptic techniques**
  - **Chapter 6: Laplace equation**
      - Laplace equation, fundamental solutions, solution of Poisson equation, Green's function on balls and half spaces, properties of harmonic functions, smoothness, mean value formula, maximum principle, Harnack inequality.
      - Comparison principle for hypoelliptic equations, subsolution and supersolutions, maximum principle for subharmonic functions and Perron's method.
      - (If time permits) Weak solutions for elliptic equations and Hilbert space methods. (if time permits) Dirichlet principle.
  - **Chapter 7: Heat equation**
      - Heat equation, fundamental solution, uniqueness, backward uniqueness, properties of Gaussian, smoothing, parabolic maximum principle, subsolution and supersolutions, Energy methods.

4 Main references

The basic references for this course would be the Lecture notes. For the most part, we would be following Evans [1]. We would sometimes follow John [3], Folland [2], Renardy-Rogers [5], Pinchover-Rubinstein [4] and some other references for particular materials as well.

Suggested books

[1] Evans, L. C. *Partial differential equations*, vol. 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[2] Folland, G. B. *Introduction to partial differential equations*, second ed. Princeton University Press, Princeton, NJ, 1995.

[3] John, F. *Partial differential equations*, fourth ed., vol. 1 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1982.

[4] Pinchover, Y., and Rubinstein, J. *An introduction to partial differential equations*. Cambridge University Press, Cambridge, 2005.

[5] Renardy, M., and Rogers, R. C. *An introduction to partial differential equations*, second ed., vol. 13 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 2004.