Introduction to Partial Differential Equations
Lecture Notes
Lecture 0

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Contents

A few words on this note

This note summarizes the results from measure and integration theory and L^p spaces that we would be using constantly throughout the course. However, the last subsection on weak convergence in L^p would not be used until much later in the course and that too only a little bit. So one can safely skip reading that subsection.

Most of the material in this note is standard and can be found in almost any book discussing measure and integration theory. You can pick your favorite among the absolute classics - Rudin [4], Folland [1], Royden [3] or Hewitt and Stromberg [2].

1 Preliminaries on Measure and Integration

1.1 Measure space

Let $(\Omega, \mathcal{M}, \mu)$ denote a measure space, i.e., Ω is a set and

- $\mathcal{M}$ is a σ-algebra on Ω, i.e., $\mathcal{M}$ is a collection of subsets of Ω such that:
  - $\emptyset \in \mathcal{M}$,
  - $A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$,
  - $\bigcup\limits_{n=1}^{\infty} A_n \in \mathcal{M}$ whenever $A_n \in \mathcal{M}$ for every n.
- μ is a measure, i.e., $\mu : \mathcal{M} \to [0, \infty]$ satisfies
  - $\mu(\emptyset) = 0$,
  - $\mu\left(\bigcup\limits_{n=1}^{\infty} A_n\right) = \bigcup\limits_{n=1}^{\infty} \mu(A_n)$, whenever $\{A_n\}$ is a disjoint countable family of members of $\mathcal{M}$ (called *μ-measurable sets*).

  Although not strictly essential, we shall also assume that
  - μ is σ-finite, i.e. there exists a countable family $\{\Omega_n\} \subset \mathcal{M}$ such that $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ and $\mu(\Omega_n) < \infty$ for every n.

The sets $E \in \mathcal{M}$ with the property that $\mu(E) = 0$ are called the μ-null sets. A measure space $(\Omega, \mathcal{M}, \mu)$ is called a *complete measure space* if any subset A of a μ-null set is in $\mathcal{M}$. We say that a property holds μ a.e. (or for μ-almost all $x \in \Omega$) if it holds everywhere on Ω except on a μ-null set. If $\Omega \subset \mathbb{R}^n$ and we do not specify the measure μ, then it is implicitly understood that μ is the Lebesgue measure on $\mathbb{R}^n$, which we define in the next subsection.

Note that if $(\Omega, \mathcal{M}, \mu)$ is a measure space and $A \in \mathcal{M}$, then the collection of sets $\mathcal{M}_A := \{A \cap B : B \in \mathcal{M}\}$ is a σ-algebra on A and $(A, \mathcal{M}_A, \mu)$ is a measure space in its own right.

1.2 Lebesgue Measure on $\mathbb{R}^n$

Lebesgue measure provides a way to talk about 'n-dimensional size' or 'volume' of reasonably 'nice' subsets of $\mathbb{R}^n$.

Theorem 1. *There exists a unique σ-algebra $\mathcal{L}$ on $\mathbb{R}^n$ and a unique measure*

$$|\cdot| : \mathcal{L} \to [0, +\infty]$$

such that

1. *Every open subset of $\mathbb{R}^n$ and hence any Borel subset of $\mathbb{R}^n$ is in $\mathcal{L}$,*
2. *For any $E \in \mathcal{L}$ and any $x \in \mathbb{R}^n$, the set $E + x := \{x + y : y \in E\}$ is also in $\mathcal{L}$ and we have*
$$|E + x| = |E| \, .$$
3. *For any ball $B \subset \mathbb{R}^n$, $|B|$ is the n-dimensional volume of the ball.*
4. *For any compact subset $K \subset \mathbb{R}^n$, $|K| < \infty$.*
5. *For every $A \in \mathcal{L}$, we have*
$$|A| = \sup \{|K| : K \subset A,\ K \text{ compact } \} = \inf \{|U| : A \subset U,\ U \text{ open } \} \, ,$$
6. *$(\mathbb{R}^n, \mathcal{L}, |\cdot|)$ is a complete measure space.*

The sets in $\mathcal{L}$ are called Lebesgue measurable sets and $|\cdot|$ is called the n-dimensional Lebesgue measure on $\mathbb{R}^n$.

1.3 Measurable and integrable functions

Definition 2. *Let $(\Omega, \mathcal{M}, \mu)$ be a measure space. A function $f : \Omega \to \mathbb{R}$ is called* μ-measurable *if*

$$f^{-1}(U) \in \mathcal{M} \text{ for every open subset } U \subset \mathbb{R}.$$

When Ω is a Lebesgue measurable subset of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$, we say f is measurable *when it is measurable with respect to the n-dimensional Lebesgue measure.*

Since all open sets are in $\mathcal{L}$, it is easy to that any continuous function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is measurable when $\Omega \subset \mathbb{R}^n$ is open. Composition of measurable functions are not always measurable. However, it is easy to see that if f is measurable and g is continuous, then $g \circ f$ is measurable.

Proposition 3. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ is measurable and $\lambda \in \mathbb{R}$. Then λf, $f + g$, fg, $\max \{f, g\}$, $\min \{f, g\}$ are all measurable. Moreover, if $f_k : \mathbb{R}^n \to \mathbb{R}$ are measurable for every $k \in \mathbb{N}$, then $\limsup\limits_{k\to\infty} f_k$ and $\liminf\limits_{k\to\infty} f_k$ are measurable.*

Definition 4. *Given a subset $A \in \mathbb{R}^n$, the function $\mathbb{1}_A : \mathbb{R}^n \to \mathbb{R}$, defined by*

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus A, \end{cases}$$

*is called the **characteristic function** or the **indicator function** of the set A.*

Proposition 5. *If $A \subset \mathbb{R}^n$ is measurable, i.e. $A \in \mathcal{L}$, then $\mathbb{1}_A$ is a measurable function.*

Definition 6. *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called* simple *if it is a* finite linear combination of indicator functions*, i.e. f is of the form*

$$f(x) = \sum_{i=1}^{m} c_i \mathbb{1}_{A_i}(x),$$

where $m \in \mathbb{N}$ and for each $1 \leq i \leq m$, we have $c_i \in \mathbb{R}$ and $A_i \in \mathcal{L}$. We define the Lebesgue integral *of a **nonnegative** simple function f as*

$$\int_{\mathbb{R}^n} f(x)\ \mathrm{d}x := \sum_{i=1}^{m} c_i |A_i|.$$

Definition 7. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a **nonnegative** measurable function. Then its* Lebesgue integral *is defined as*

$$\int_{\mathbb{R}^n} f(x)\ \mathrm{d}x := \sup \left\{ \int_{\mathbb{R}^n} \psi(x)\ \mathrm{d}x : 0 \leq \psi \leq f,\ \psi \text{ simple} \right\}.$$

Definition 8. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Then the nonnegative measurable functions $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ are called the positive and negative parts of f, respectively. If at least one of these functions has a **finite** integral, then we define the* Lebesgue integral *of f as*

$$\int_{\mathbb{R}^n} f(x)\ \mathrm{d}x := \int_{\mathbb{R}^n} f^+(x)\ \mathrm{d}x - \int_{\mathbb{R}^n} f^-(x)\ \mathrm{d}x.$$

For any $A \in \mathcal{L}$, we define the integral of f in A *as*

$$\int_A f(x)\ \mathrm{d}x := \int_{\mathbb{R}^n} \mathbb{1}_A(x) f(x)\ \mathrm{d}x.$$

Definition 9. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. We say f is **integrable**, written as $f \in L^1(\mathbb{R}^n)$ if*

$$\int_{\mathbb{R}^n} |f(x)|\ \mathrm{d}x < \infty.$$

*For any $\Omega \in \mathcal{L}$, we say f is **integrable in** Ω if*

$$\int_{\Omega} |f(x)|\ \mathrm{d}x < \infty.$$

For such functions, we shall use the notations

$$\|f\|_{L^1(\Omega)} = \|f\|_{L^1} = \int_\Omega |f| \ \mathrm{d}\mu = \int |f|.$$

We also identify two function which coincides a.e.

Definition 10. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. We say f is* ***locally integrable****, written as $f \in L^1_{loc}(\mathbb{R}^n)$ if f is integrable in K for any* ***compact*** *subset $K \subset \mathbb{R}^n$.*

2 Basic results in Lebesgue integration

2.1 Convergence theorems

Theorem 11 (Monotone convergence theorem)**.** *Let $\{f_k\}$ be a sequence of non-negative measurable functions in $\Omega \subset \mathbb{R}^n$ that satisfy*

$$0 \le f_1 \le f_2 \le \ldots \le f_k \le f_{k+1} \le \ldots \ \text{a.e. in } \Omega.$$

Define the function f by

$$f(x) := \lim_{k\to\infty} f_k(x) \qquad \text{for a.e. } x \in \Omega.$$

Then we have

$$\int_\Omega f(x) \ \mathrm{d}x = \lim_{k\to\infty} \int_\Omega f_k(x) \ \mathrm{d}x.$$

Note that integrals in the last equality can be both infinite. A simple corollary is the following.

Corollary 12. *Let $\{f_k\}$ be a sequence of nonnegative functions in $L^1(\Omega)$ that satisfy*

- $0 \le f_1 \le f_2 \le \ldots \le f_k \le f_{k+1} \le \ldots$ *a.e. on* Ω,
- $\sup_k f_k < \infty$.

Then $f_k(x)$ converges a.e. on Ω to a finite limit, which we denote by $f(x)$. The function f belongs to $L^1(\Omega)$ and $\|f_k - f_1\|_{L^1} \to 0$.

Theorem 13 (Fatou's lemma)**.** *Let $\{f_k\}$ be a sequence of nonnegative measurable functions in $\Omega \subset \mathbb{R}^n$. Define the function f by*

$$f(x) := \liminf_{k\to\infty} f_k(x) \qquad \text{for a.e. } x \in \Omega.$$

Then we have

$$\int_\Omega f(x) \ \mathrm{d}x \le \liminf_{k\to\infty} \int_\Omega f_k(x) \ \mathrm{d}x.$$

An easy corollary is

Corollary 14. *Let $\{f_k\}$ be a sequence of functions in $L^1(\Omega)$ that satisfy*

- *for all k, $f_k \geq 0$ a.e.*
- $\sup_k \int_\Omega f_k < \infty$.

For a.e. $x \in \Omega$, we set $f(x) = \liminf_{k\to\infty} f_k(x) \leq +\infty$. Then $f \in L^1(\Omega)$ and

$$\int_\Omega f(x)\ \mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega f_k\ \mathrm{d}x.$$

Theorem 15 (Dominated convergence theorem)**.** *Let $\{f_k\}$ be a sequence of functions in $L^1(\Omega)$ that satisfy*

- *$f_k(x) \to f(x)$ a.e. on Ω,*
- *there exists a function $g \in L^1(\Omega)$ such that for all k, we have $|f_k(x)| \leq |g(x)|$ a.e. on Ω.*

Then $f \in L^1(\Omega)$ and $\|f_k - f_1\|_{L^1} \to 0$.

2.2 Change of variable

We denote the set of all $n \times n$ invertible real matrices by $\mathbb{GL}(n, \mathbb{R})$. With an abuse of notation, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is an invertible linear transformation, we also write $T \in \mathbb{GL}(n, \mathbb{R})$.

Theorem 16. *Let $T \in \mathbb{GL}(n, \mathbb{R})$.*

1. *If $A \in \mathcal{L}$, then $T(A) \in \mathcal{L}$ and $|T(A)| = |\det T|\,|A|$.*
2. *If $f : \mathbb{R}^n \to \mathbb{R}$ is measurable, then so is $f \circ T$. If $f \geq 0$ or $f \in L^1(\mathbb{R}^n)$, then we have*

$$\int_{\mathbb{R}^n} f(x)\ \mathrm{d}x = |\det T| \int_{\mathbb{R}^n} f(Tx)\ \mathrm{d}x.$$

Theorem 17. *Let $\Omega \subset \mathbb{R}^n$ be open and let $\Phi : \Omega \to \mathbb{R}^n$ be a C^1 diffeomorphism.*

1. *If $A \subset \Omega$ is measurable, then so is $\Phi(A)$ and*

$$|\Phi(A)| = \int_A |\det D\Phi|(x)\ \mathrm{d}x.$$

2. *If $f : \Omega \to \mathbb{R}$ is measurable, then so is $f \circ \Phi$. If $f \geq 0$ or $f \in L^1(\Omega)$, then we have*

$$\int_{\Phi(\Omega)} f(x)\ \mathrm{d}x = \int_\Omega f \circ \Phi(x)\,|\det D\Phi|(x)\ \mathrm{d}x.$$

2.3 Product measures

Let $(\Omega_1, \mathcal{M}_1, \mu_1)$ and $(\Omega_2, \mathcal{M}_2, \mu_2)$ be two measure spaces that are σ-finite. One can then define in a standard way the structure of a *complete* measure space $(\Omega, \mathcal{M}, \mu)$ on the Cartesian product $\Omega = \Omega_1 \times \Omega_2$. The product measure so obtained is often denoted by $\mu = \mu_1 \times \mu_2$.

There are two cases of product measures that are of particular interest to us. The first is the obvious one. It is suggested both by the fact that for any two $m, n \in \mathbb{N}$, we have $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ and also by the heuristic notions like 'area = length × width', 'volume = cross-sectional area × height = length × width × height' etc. For stating the result, we would switch to writing μ_n as the n-dimensional Lebesgue measure and $\mathcal{L}_n$ as the Lebesgue σ-algebra on $\mathbb{R}^n$.

Proposition 18. *The complete measure space on the Cartesian product corresponding to the measure spaces* $(\mathbb{R}^m, \mathcal{L}_m, \mu_m)$ *and* $(\mathbb{R}^n, \mathcal{L}_n, \mu_n)$ *is precisely* $(\mathbb{R}^{m+n}, \mathcal{L}_{m+n}, \mu_{m+n})$.

The next important product measure situation is the polar co-ordinate formula. This essentially depends on two facts. The first is that the Lebesgue measure of a point is zero and thus for integration, $\mathbb{R}^n$ and $\mathbb{R}^n \setminus \{0\}$ are the same. The second fact is that $\mathbb{R}^n \setminus \{0\}$ can be bijectively mapped into a Cartesian product. Let $\mathbb{S}^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere. For $x \in \mathbb{R}^n \setminus \{0\}$, let us use the notation

$$\hat{x} = \frac{x}{|x|} \in \mathbb{S}^{n-1}.$$

Then the map

$$\Phi : \mathbb{R}^n \setminus \{0\} \to (0, \infty) \times \mathbb{S}^{n-1}$$

given by

$$\Phi(x) = (|x|, \hat{x})$$

is a continuous bijection with a continuous inverse. Thus, the n-dimensional Lebesgue measure μ_n on $\mathbb{R}^n$ (more precisely, its restriction to $\mathbb{R}^n \setminus \{0\}$) induces a complete measure m on the Cartesian product $(0, \infty) \times \mathbb{S}^{n-1}$ in a natural way.[1] Now, as one can guess, there exist two essentially unique measures ρ and Σ on $(0, \infty)$ and $\mathbb{S}^{n-1}$ respectively, so that $m = \rho \times \Sigma$. In fact, by scaling properties of μ_n, one can show that ρ can only be given by

$$\rho(A) = \int_A r^{n-1} \, \mathrm{d}r,$$

where, as the notation indicates, the integration is performed with respect to the one dimensional Lebesgue measure. Thus we have

[1] One way to do it is by defining

$$m(A) = \mu_n\left(\Phi^{-1}(A)\right) \text{ for all Borel sets } A$$

and then completing the Borel measure so obtained.

Proposition 19 (Polar coordinate formula)**.** *There exists a complete measure* Σ *(essentially unique) on* $\mathbb{S}^{n-1}$ *such that for any* $f : \mathbb{R}^n \to \mathbb{R}$ *measurable with either* f *is nonnegative or* $f \in L^1(\mathbb{R}^n)$, *we have*

$$\int_{\mathbb{R}^n} f(x) \ \mathrm{d}x = \int_0^\infty \int_{\mathbb{S}^{n-1}} f(ry) \, r^{n-1} \ \mathrm{d}\Sigma(y) \ \mathrm{d}r.$$

Now we turn to the question of when we can permute the order of the integrals.

Theorem 20 (Fubini theorem)**.** *Let* $(\Omega_1, \mathcal{M}_1, \mu_1)$ *and* $(\Omega_2, \mathcal{M}_2, \mu_2)$ *be two complete measure spaces. Let* $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ *be integrable in* $\Omega = \Omega_1 \times \Omega_2$. *Then*

$$\begin{aligned} x &\mapsto f(x,y) \quad \textit{is } \mu_1\textit{-integrable in } \Omega_1 \textit{ for } \mu_2\textit{-almost all } y \in \Omega_2, \\ y &\mapsto f(x,y) \quad \textit{is } \mu_2\textit{-integrable in } \Omega_2 \textit{ for } \mu_1\textit{-almost all } x \in \Omega_1. \end{aligned}$$

Furthermore,

$$\begin{aligned} x &\mapsto \int_{\Omega_2} f(x,y) \ \mathrm{d}\mu_2 \quad \textit{is } \mu_1\textit{-integrable in } \Omega_1, \\ y &\mapsto \int_{\Omega_1} f(x,y) \ \mathrm{d}\mu_1 \quad \textit{is } \mu_2\textit{-integrable in } \Omega_2 \end{aligned}$$

and we have the equality

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} f(x,y) \ \mathrm{d}(\mu_1 \times \mu_2) &= \int_{\Omega_1} \left(\int_{\Omega_2} f(x,y) \ \mathrm{d}\mu_2 \right) \ \mathrm{d}\mu_1 \\ &= \int_{\Omega_2} \left(\int_{\Omega_1} f(x,y) \ \mathrm{d}\mu_1 \right) \ \mathrm{d}\mu_2. \end{aligned} \tag{1}$$

Theorem 21 (Tonelli theorem)**.** *Let* $(\Omega_1, \mathcal{M}_1, \mu_1)$ *and* $(\Omega_2, \mathcal{M}_2, \mu_2)$ *be two complete* σ-finite *measure spaces. Let* $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ *be a nonnegative measurable function. Then* (1) *holds and thus the individual integrals are either all finite or all infinite.*

Combining the two results above, we obtain a useful version which is usually called the Fubini-Tonelli theorem.

Theorem 22 (Fubini-Tonelli theorem)**.** *Let* $(\Omega_1, \mathcal{M}_1, \mu_1)$ *and* $(\Omega_2, \mathcal{M}_2, \mu_2)$ *be two complete* σ-finite *measure spaces. Let* $f : \Omega_1 \times \Omega_2 \to \mathbb{R}$ *be a measurable function. Then*

$$\begin{aligned} \int_{\Omega_1 \times \Omega_2} |f(x,y)| \ \mathrm{d}(\mu_1 \times \mu_2) &= \int_{\Omega_1} \left(\int_{\Omega_2} |f(x,y)| \ \mathrm{d}\mu_2 \right) \ \mathrm{d}\mu_1 \\ &= \int_{\Omega_2} \left(\int_{\Omega_1} |f(x,y)| \ \mathrm{d}\mu_1 \right) \ \mathrm{d}\mu_2. \end{aligned} \tag{2}$$

Further, if any one of the integrals is finite, so are the other two and we have the following equality between finite signed integrals

$$\int_{\Omega_1 \times \Omega_2} f(x,y)\ \mathrm{d}\left(\mu_1 \times \mu_2\right) = \int_{\Omega_1} \left(\int_{\Omega_2} f(x,y)\ \mathrm{d}\mu_2 \right)\ \mathrm{d}\mu_1$$
$$= \int_{\Omega_2} \left(\int_{\Omega_1} f(x,y)\ \mathrm{d}\mu_1 \right)\ \mathrm{d}\mu_2. \qquad (3)$$

2.4 Differentiation

An integrable function is 'approximately continuous' at almost every point.

Theorem 23 (Lebesgue differentiation theorem)**.** *Let* $f \in L^1_{loc}\left(\mathbb{R}^n\right)$. *Then for a.e.* $x_0 \in \mathbb{R}^n$, *we have*

$$⨍_{B(x_0,r)} f\left(x\right)\ \mathrm{d}x := \frac{1}{\left|B\left(x_0,r\right)\right|} \int_{B(x_0,r)} f\left(x\right)\ \mathrm{d}x \to f\left(x_0\right) \qquad \text{as } r \to 0+.$$

Moreover, for a.e. $x_0 \in \mathbb{R}^n$,

$$⨍_{B(x_0,r)} \left|f\left(x\right) - f\left(x_0\right)\right|\ \mathrm{d}x \to 0 \qquad \text{as } r \to 0+.$$

For a given f, the points $x_0 \in \mathbb{R}^n$ for which the conclusions of the theorem holds are called **Lebesgue points of** f.

3 L^p spaces

3.1 Support of a continuous function

Definition 24. *Let* $f : \mathbb{R}^n \to \mathbb{R}$ *be continuous. Then the* support of f, *denoted by* $\operatorname{supp} f$, *is defined as the closure of the set where* f *is nonzero. More precisely,*

$$\operatorname{supp} f := \overline{\left\{x \in \mathbb{R}^n : f\left(x\right) \neq 0\right\}}.$$

The definition of support for functions defined on an open proper subset of of $\mathbb{R}^n$ is superficially the same, but has an important difference.

Definition 25. *Let* $\Omega \subset \mathbb{R}^n$ *be open and let* $f : \Omega \to \mathbb{R}$ *be continuous. Then the* support of f, *denoted by* $\operatorname{supp} f$, *is defined as the closure in the* ***subspace topology of*** Ω *of the set where* f *is nonzero. More precisely,*

$$\operatorname{supp} f := \overline{\left\{x \in \Omega : f\left(x\right) \neq 0\right\}}^{\Omega},$$

Note that when Ω is a proper subset of $\mathbb{R}^n$, there is a crucial difference from the earlier definition. Since Ω is always closed in the subspace topology of Ω, the support of any function which vanish nowhere in Ω is Ω itself and is thus **not a closed set** $\mathbb{R}^n$.

Definition 26. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be continuous. f is said to be* compactly supported*, denoted $f \in C_c(\mathbb{R}^n)$ if* $\operatorname{supp} f$ *is a compact subset of $\mathbb{R}^n$. If f is moreover C^k for some $k \in \mathbb{N}$ or C^∞ and has compact support, we write $f \in C_c^k(\mathbb{R}^n)$ or $f \in C_c^\infty(\mathbb{R}^n)$, respectively.*

Similarly, if $\Omega \subset \mathbb{R}^n$ is open and let $f : \Omega \to \mathbb{R}$ is continuous, then f is said to be compactly supported in Ω*, denoted $f \in C_c(\Omega)$ if* $\operatorname{supp} f$ *is a compact subset of $\mathbb{R}^n$. As before, if f is moreover C^k for some $k \in \mathbb{N}$ or C^∞ and has compact support in Ω, we write $f \in C_c^k(\Omega)$ or $f \in C_c^\infty(\Omega)$, respectively.*

Although the definition for Ω and $\mathbb{R}^n$ looks exactly the same, since the definition of support differes, once again there is a crucial difference when Ω is a proper subset of $\mathbb{R}^n$. Since $\Omega \subset \mathbb{R}^n$ is open, one can deduce that if f is compactly supported in Ω, the support of f is a **closed set** in $\mathbb{R}^n$ and hence must be strictly contained in Ω. Hence f must vanish not only on $\partial\Omega$, but in a neighborhood of $\partial\Omega$ as well.

3.2 Basic definitions and properties of L^p spaces

Definition 27. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq p \leq \infty$. We say that a measurable function $u : \Omega \to \mathbb{R}$ belongs to $L^p(\Omega)$ if*

$$\|u\|_{L^p} = \begin{cases} \left(\int_\Omega |u(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty \\ \inf \{ \alpha : |u(x)| \leq \alpha \text{ a.e. in } \Omega \} & \text{if } p = \infty \end{cases}$$

is finite. As above, if $u : \Omega \to \mathbb{R}^N$, $u = \left(u^1, \cdots, u^N\right)$, is such that $u^i \in L^p(\Omega)$, for every $i = 1, \cdots, N$, we write $u \in L^p\left(\Omega; \mathbb{R}^N\right)$.

Definition 28. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq p < \infty$. We say that a measurable function $u : \Omega \to \mathbb{R}$ belongs to $L^p_{loc}(\Omega)$ if we have*

$$\int_\Omega |u(x)|^p \, \mathrm{d}x < \infty \qquad \text{for any compact } K \subset \Omega.$$

In the following, we let p' be the *conjugate exponent* of p. It is defined by

$$\frac{1}{p} + \frac{1}{p'} = 1 \quad \Leftrightarrow \quad p' = \frac{p}{p-1}$$

with the convention that if $p = 1$, respectively $p = \infty$, then $p' = \infty$, respectively $p' = 1$. Now we summarize the most important properties of L^p spaces that we need.

Theorem 29. *Let $\Omega \subset \mathbb{R}^n$ be open and $1 \leq p \leq \infty$.*

(i) $\|\cdot\|_{L^p}$ is a norm and $L^p(\Omega)$, equipped with this norm, is a Banach space[2].

[2] A **Banach space** is a normed linear space which is complete as a metric space (with the metric induced from the norm).

The space $L^2(\Omega)$ is a Hilbert space[3] *with inner product given by*

$$\langle u; v\rangle = \int_\Omega u(x)\, v(x)\, dx.$$

(ii) ***Hölder inequality*** *asserts that if $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ where $1/p + 1/p' = 1$, then $uv \in L^1(\Omega)$ and moreover*

$$\|uv\|_{L^1} \le \|u\|_{L^p} \|v\|_{L^{p'}}.$$

(iii) ***Young's inequality*** *asserts that if $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ where $1/p + 1/p' = 1$, then $uv \in L^1(\Omega)$ and moreover*

$$\|uv\|_{L^1} \le \frac{1}{p}\|u\|_{L^p}^p + \frac{1}{p'}\|v\|_{L^{p'}}^{p'}.$$

(iv) ***Young's inequality with*** ε *asserts that for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that for any $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$ where $1/p + 1/p' = 1$, we have*

$$\|uv\|_{L^1} \le \varepsilon\|u\|_{L^p}^p + C_\varepsilon\|v\|_{L^{p'}}^{p'}.$$

(v) ***Minkowski inequality*** *asserts that*

$$\|u + v\|_{L^p} \le \|u\|_{L^p} + \|v\|_{L^p}.$$

(vi) ***Riesz theorem****: the dual space of L^p, denoted by $(L^p)'$, can be identified with $L^{p'}(\Omega)$ where $1/p + 1/p' = 1$ provided $1 \le p < \infty$. More precisely, if $\varphi \in (L^p)'$ with $1 \le p < \infty$, then there exists a unique $u \in L^{p'}$ so that*

$$\langle \varphi; f\rangle = \varphi(f) = \int_\Omega u(x)\, f(x)\, dx, \quad \forall f \in L^p(\Omega)$$

and moreover

$$\|u\|_{L^{p'}} = \|\varphi\|_{(L^p)'}.$$

As a consequence, for $1 \le p < \infty$, there is an alternative characterization of the L^p norm.

$$\|u\|_{L^p} = \sup_{\substack{v \in L^{p'},\\ \|v\|_{L^{p'}} \le 1}} \int uv = \sup_{\substack{v \in L^{p'},\\ \|v\|_{L^{p'}} = 1}} \int uv.$$

(vii) L^p is separable[4] *if $1 \le p < \infty$ and reflexive if $1 < p < \infty$.*

[3] A **Hilbert space** is an inner product space which is complete as a metric space (with the metric induced from the inner product).

[4] A topological space is called **separable** if it has a countable dense subset.

(viii) *Let $1 \leq p < \infty$. The piecewise constant functions (also called step functions if $\Omega \subset \mathbb{R}$) or the $C_c^\infty(\Omega)$ functions are dense in L^p. More precisely, if $u \in L^p(\Omega)$ then there exist $u_\nu \in C_c^\infty(\Omega)$ (or u_ν piecewise constant) so that*

$$\lim_{\nu \to \infty} \|u_\nu - u\|_{L^p} = 0.$$

Remark 30. ***(i)*** *In the case $p = 2$ and hence $p' = 2$, Hölder inequality is nothing but* Cauchy-Schwarz *inequality*

$$\|uv\|_{L^1} \leq \|u\|_{L^2} \|v\|_{L^2}, \quad \text{i.e.} \int_\Omega |uv| \leq \left(\int_\Omega u^2\right)^{1/2} \left(\int_\Omega v^2\right)^{1/2}.$$

(ii) *In Riesz theorem the result is false if $p = \infty$ (and hence $p' = 1$).*

(iii) *In the following, we always make the identification $(L^p)' = L^{p'}$. Summarizing the results on duality we have*

$$(L^p)' = L^{p'} \quad \text{if } 1 < p < \infty,$$
$$\left(L^2\right)' = L^2, \quad \left(L^1\right)' = L^\infty, \quad L^1 \underset{\neq}{\subset} (L^\infty)'.$$

(iv) *The meaning of L^p* reflexive *is that the bidual of L^p, $(L^p)''$, can be identified with L^p.*

(v) *The last statement in the theorem is false if $p = \infty$.*

3.3 Convolution

Definition 31. *Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be measurable functions. Then the* convolution of f and g, *denoted $f * g$, is defined as*

$$f * g(x) := \int_{\mathbb{R}^n} f(x - y) g(y) \, dy,$$

for all $x \in \mathbb{R}^n$ for which the integral exists.

Proposition 32. *Let $f, g, h : \mathbb{R}^n \to \mathbb{R}$ be measurable functions. Assuming all the integrals in question exists, we have*

1. $f * g = g * f$.
2. $(f * g) * h = f * (g * h)$.
3. $\operatorname{supp}(f * g) \subset \overline{\operatorname{supp} f + \operatorname{supp} g}$, *where the sum on the right means the set*

$$\{x + y : x \in \operatorname{supp} f, \ y \in \operatorname{supp} g\}.$$

Theorem 33 (Young's inequality for convolutions)**.** *Let $1 \leq p, q, r \leq \infty$ such that*

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1.$$

If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then $f * g \in L^r(\mathbb{R}^n)$ and we have

$$\|f * g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}.$$

Proposition 34. *If* $f \in L^1(\mathbb{R}^n)$ *and* $g \in C^k(\mathbb{R}^n)$ *with* $D^\alpha g \in L^\infty(\mathbb{R}^n)$ *for each multiindex* α *with* $|\alpha| \leq k$*, then* $f * g \in C^k(\mathbb{R}^n)$ *and we have*

$$D^\alpha (f * g) = f * D^\alpha g$$

for each multiindex α *with* $|\alpha| \leq k$*.*

Proposition 35. *If* $f \in L^1_{loc}(\mathbb{R}^n)$ *and* $g \in C^k_c(\mathbb{R}^n)$ *then* $f * g \in C^k(\mathbb{R}^n)$ *and we have*

$$D^\alpha (f * g) = f * D^\alpha g$$

for each multiindex α *with* $|\alpha| \leq k$*.*

3.4 Mollifiers

Definition 36. *A function* $\phi : \mathbb{R}^n \to \mathbb{R}$ *is called a* ***smoothing kernel*** *or a* ***mollifying kernel*** *if*

(i) $\phi \in C^\infty_c(\mathbb{R}^n)$,

(ii) $\operatorname{supp} \phi \subset \overline{B(0,1)}$,

(iii) $\phi \geq 0$ *and* $\phi \not\equiv 0$,

(iv) For some finite nonzero real number $a \in \mathbb{R}$ *with* $a > 0$*, we have*

$$\int_{\mathbb{R}^n} \phi(x)\ \mathrm{d}x = a.$$

Given a smoothing kernel ϕ*, the* ***sequence of mollifiers*** *corresponding to* ϕ *is the sequence* $\{\phi_k\}_{k\in\mathbb{N}} \subset C^\infty_c(\mathbb{R}^n)$ *where*

$$\phi_k(x) = \frac{1}{a} k^n \phi(kx) \qquad \textit{for every } x \in \mathbb{R}^n$$

for every $k \in \mathbb{N}$*.*

A typical example of such a smoothing kernel is provided by the function

$$\phi(x) = \begin{cases} e^{-\frac{1}{(1-|x|^2)}} & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 1. \end{cases}$$

Theorem 37. *Let* $\{\phi_k\}_{k\in\mathbb{N}} \subset C^\infty_c(\mathbb{R}^n)$ *be a sequence of mollifiers. Then*

(i) Let $1 \leq p < \infty$ *and let* $f \in L^p(\mathbb{R}^n)$*. Then* $f_k := \phi_k * f \in C^\infty(\mathbb{R}^n)$ *for every* $k \in \mathbb{N}$ *and we have*

$$\|f_k - f\|_{L^p(\mathbb{R}^n)} \to 0 \qquad \textit{as } k \to \infty.$$

(ii) *Let $f \in C(\mathbb{R}^n)$. Then $f_k := \phi_k * f \in C^\infty(\mathbb{R}^n)$ for every $k \in \mathbb{N}$ and*

$$f_k \to f \quad \text{uniformly} \quad \text{on any compact subset } K \subset \mathbb{R}^n.$$

(iii) *Let $f \in C^k(\mathbb{R}^n)$ for some $k \in \mathbb{N}$. Then $f_s := \phi_s * f \in C^\infty(\mathbb{R}^n)$ for every $s \in \mathbb{N}$ and*

$$D^\alpha f_s \to D^\alpha f \quad \text{uniformly} \quad \text{on any compact subset } K \subset \mathbb{R}^n$$

for any multiindex α with $|\alpha| \leq k$.

The process of constructing the sequence of functions $f_k := \phi_k * f$, where $\{\phi_k\}_{k\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ is a sequence of mollifiers, is called *mollifying* f or simply *mollification*.

3.5 Weak convergence in L^p

We now turn our attention to the notions of convergence in L^p spaces. The natural notion, called *strong convergence*, is the one induced by the $\|\cdot\|_{L^p}$ norm. We also often need a weaker notion of convergence known as *weak convergence*. We now define these notions.

Definition 38. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $1 \leq p \leq \infty$.*

(i) *A sequence u_ν is said to (strongly)* converge *to u in L^p if u_ν, $u \in L^p$ and if*

$$\lim_{\nu\to\infty} \|u_\nu - u\|_{L^p} = 0.$$

We denote this convergence by $u_\nu \to u$ in L^p.

(ii) *If $1 \leq p < \infty$, we say that the sequence u_ν* weakly converges *to u in L^p if u_ν, $u \in L^p$ and if*

$$\lim_{\nu\to\infty} \int_\Omega [u_\nu(x) - u(x)]\, \varphi(x)\, dx = 0, \quad \forall\, \varphi \in L^{p'}(\Omega).$$

This convergence is denoted by $u_\nu \rightharpoonup u$ in L^p.

(iii) *If $p = \infty$, the sequence u_ν is said to* weak $*$ converge *to u in L^∞ if u_ν, $u \in L^\infty$ and if*

$$\lim_{\nu\to\infty} \int_\Omega [u_\nu(x) - u(x)]\, \varphi(x)\, dx = 0, \quad \forall\, \varphi \in L^1(\Omega)$$

and is denoted by: $u_\nu \overset{*}{\rightharpoonup} u$ in L^∞.

(iv) *Let $\{u_\nu\}$ be a sequence of measurable functions on Ω into $\mathbb{R}$ and let $u : \Omega \to \mathbb{R}$ be a measurable function in Ω. The sequence $\{u_\nu\}$* converges in measure *to u , if for any $\varepsilon > 0$*

$$\lim \mu\left(\{x \in \Omega : |u_\nu(x) - u(x)| > \varepsilon\}\right) = 0.$$

Remark 39. ***(i)*** *We speak of weak $*$ convergence in L^∞ instead of weak convergence, because as seen above the dual of L^∞ is strictly larger than L^1. Formally, however, weak convergence in L^p and weak $*$ convergence in L^∞ take the same form.*

(ii) *The limit (weak or strong or in measure) is unique.*

(iii) *It is obvious that*

$$u_\nu \to u \text{ in } L^p \quad \Rightarrow \quad \begin{cases} u_\nu \rightharpoonup u \text{ in } L^p & \text{if } 1 \le p < \infty \\ u_\nu \overset{*}{\rightharpoonup} u \text{ in } L^\infty & \text{if } p = \infty. \end{cases}$$

If Ω has finite measure, then $u_\nu \to u$ a.e. implies $u_\nu \to u$ in measure. If $u_\nu \to u$ in measure, then there is a subsequence u_{ν_i} that converges to u a.e.

Theorem 40. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. The following properties then hold.*

(i) *If $u_\nu \overset{*}{\rightharpoonup} u$ in L^∞, then $u_\nu \rightharpoonup u$ in L^p, $\forall\, 1 \le p < \infty$.*

(ii) *If $1 \le p \le \infty$ and $u_\nu \to u$ in L^p, then*

$$\|u\|_{L^p} = \lim_{\nu \to \infty} \|u_\nu\|_{L^p} \ .$$

(iii) *If $1 \le p < \infty$ and if $u_\nu \rightharpoonup u$ in L^p, then there exists a constant $\gamma > 0$ so that*

$$\|u_\nu\|_{L^p} \le \gamma \quad \text{and} \quad \|u\|_{L^p} \le \liminf_{\nu \to \infty} \|u_\nu\|_{L^p} \ .$$

The result remains valid if $p = \infty$ and if $u_\nu \overset{*}{\rightharpoonup} u$ in L^∞.

(iv) *If $1 < p < \infty$ and if there exists a constant $\gamma > 0$ so that $\|u_\nu\|_{L^p} \le \gamma$, then there exist a subsequence $\{u_{\nu_i}\}$ and $u \in L^p$ so that*

$$u_{\nu_i} \rightharpoonup u \quad \text{in } L^p.$$

The result remains valid if $p = \infty$, the conclusion is then $u_{\nu_i} \overset{*}{\rightharpoonup} u$ in L^∞.

(v) *Let $1 \le p \le \infty$ and $u_\nu \to u$ in L^p, then there exist a subsequence $\{u_{\nu_i}\}$ and $h \in L^p$ such that*

$$u_{\nu_i} \to u \text{ a.e.} \quad \text{and} \quad |u_{\nu_i}| \le h \text{ a.e.}$$

Remark 41. ***(i)*** *Comparing (ii) and (iii) of the theorem, we see that weak convergence ensures the lower semicontinuity of the norm, while strong convergence guarantees its continuity.*

(ii) *The most interesting part of the theorem is (iv). We know that in $\mathbb{R}^n$, Bolzano-Weierstrass theorem ascertains that from any bounded sequence we can extract a convergent subsequence. This is false in L^p spaces (and more generally in infinite dimensional spaces); but it is true if we replace strong convergence by weak convergence.*

(iii) *The result (iv) is, however, false if $p = 1$; this is a consequence of the fact that L^1 is not a reflexive space. To deduce, up to the extraction of a subsequence, weak convergence, it is not sufficient to have $\|u_\nu\|_{L^1} \le \gamma$, we need a condition known as "equiintegrability".*

Part (ii) has a partial converse.

Theorem 42 (Radon-Riesz)**.** *Let* $1 < p < \infty$ *and let* $u_n : \Omega \to \mathbb{R}$ *be a sequence in* $L^p(\Omega)$ *converging weakly to* $u \in L^p(\Omega)$ *and*

$$\|u_n\|_{L^p(\Omega)} \to \|u\|_{L^p(\Omega)} .$$

Then

$$u_n \to u \qquad \text{in } L^p(\Omega) .$$

There is another converse which works even when $p = 1$.

Theorem 43. *Let* $1 \leq p < \infty$ *and let* $u_n : \Omega \to \mathbb{R}$ *be a sequence in* $L^p(\Omega)$ *such that* $u_n \to u$ *a.e and*

$$\|u_n\|_{L^p(\Omega)} \to \|u\|_{L^p(\Omega)} .$$

Then

$$u_n \to u \qquad \text{in } L^p(\Omega) .$$

We now turn to Riemann-Lebesgue theorem which allows one to easily construct weakly convergent sequences that do not converge strongly. This theorem is particularly useful when dealing with *Fourier series* (there $u(x) = \sin x$ or $\cos x$).

Theorem 44 (Riemann-Lebesgue theorem)**.** *Let* $1 \leq p \leq \infty$ *and* $u \in L^p(\Omega)$ *where* $\Omega = \prod_{i=1}^{n} (a_i, b_i)$. *Let* u *be extended by periodicity from* Ω *to* $\mathbb{R}^n$ *and define*

$$u_\nu(x) = u(\nu x) \quad \text{and} \quad \overline{u} = \frac{1}{|\Omega|} \int_\Omega u(x)\, dx$$

then $u_\nu \rightharpoonup \overline{u}$ *in* L^p *if* $1 \leq p < \infty$ *and, if* $p = \infty$, $u_\nu \overset{*}{\rightharpoonup} \overline{u}$ *in* L^∞.

Suggested books

[1] Folland, G. B. *Real analysis*, second ed. Pure and Applied Mathematics (New York). John Wiley & Sons, Inc., New York, 1999. Modern techniques and their applications, A Wiley-Interscience Publication.

[2] Hewitt, E., and Stromberg, K. *Real and abstract analysis*. Graduate Texts in Mathematics, No. 25. Springer-Verlag, New York-Heidelberg, 1975. A modern treatment of the theory of functions of a real variable, Third printing.

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[4] Rudin, W. *Real and complex analysis*, third ed. McGraw-Hill Book Co., New York, 1987.