Introduction to Partial Differential Equations

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1 Course Objective

To provide a thorough introduction to the theoretical/analytical methods in Partial Differential Equations.

2 Prerequisites

- **Analysis** A course in Real analysis and Measure and Integration is crucial. These would be used quite freely.
- **ODE** The Cauchy-Peano-Picard-Lindelöf-Lischitz theory of existence of solutions to Ordinary Differntial Equations and solutions of simple ODEs would be used in the course.
- **Functional analysis** A first course in functional analysis would be needed in the last half of the course.
- **Function spaces** L^p spaces would be used constantly.

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3 Course contents and outline

Our goal is to cover the following topics.

- **Part 1: Introductory material**
  - **Chapter 1: Introduction** Multiindex notation, Examples of different types of PDEs, linear and nonlinear equations, equations and systems. Hadamard's well-posedness notion. Different types of problems for PDEs, Cauchy problem or IVP, BVP, mixed problems. Brief idea of weak solutions in different contexts.
  - **Chapter 2: Separation of Variables method** Fourier series, Separation of variables method for the heat equation, one dimensional wave equation and the Laplace equation : formal solution, convergence and regularity.
  - **Chapter 3: Basic behavior of Linear differential operators**
    - Fourier transform in L^1 and L^2. Basic properties, Differentiation to multiplication, convolutions to products, Fourier inversion formula.
    - Basic behavior of linear PDEs: Symbol of a PDE, Characteristic variety, Cauchy problem and noncharacteristic hypersurfaces, Ellipticity and ill-posedness of the Cauchy problem, Cauchy problem and characteristic curves, Hyperbolicity.
    - Idea of a Fundamental solution: Mention of the Ehrenpreis-Malgrange theorem (No proof), Hypoellipticity and analytic-hypoellipticity, parametrix.
- **Part 2: PDEs as propagation: Hyperbolic techniques**
  - **Chapter 4: The Wave equation**
    - Constant coefficient linear Transport equation
    - Wave equation in one space dimension, D'Alembert's formula
    - Wave equation in higher dimensions
      - Spherical means, Euler-Poisson-Darboux equation,
      - Solution for $n = 3$, Kirchoff formula.
      - Hadamard method of descent and solution for $n = 2$ Poisson formula.
      - Solution for odd n and even n.
      - Energy methods, uniqueness, Domain of dependence, Huygen's principle, finite speed of propagation.
  - **Chapter 5: Nonlinear First order Equations**
    - Geometric of a first order PDE, integral manifold
    - Brief treatment of Contact geometry and differential forms

    * Method of characteristics, Characteristic characteristic ODEs, examples of solution by method of characteristic for linear and quasilinear equations
    * Fully nonlinear equations, eikonal equation and Huygen's principle.
    * Hamilton-Jacobi equations, the need for weak solutions, connection with Calculus of Variations and Hopf-Lax formula.
    * Shocks and rarefaction in nonlinear conservation laws, weak solutions, Rankine-Hugoniot jump conditions, Lax entropy conditions, Lax-Oleinik formula.

- **Part 3: Elliptic and hypoelliptic techniques**
  - **Chapter 7: Laplace equation**
    * Fundamental solutions, solution of Poisson equation, Green's function on balls and half spaces, properties of harmonic functions, mean value formula, maximum principle, Harnack inequality, regularity estimates.
    * Comparison principle for hypoelliptic equations, subsolution and supersolutions, maximum principle for subharmonic functions and Perron's method.
    * Weak derivatives and Sobolev spaces (basic idea), Poincaré inequality, existence of weak solutions for the Dirichlet problem, Hilbert space method and Dirichlet integral method, regularity.
  - **Chapter 8: Heat equation**
    * Fundamental solution, properties of Gaussian, parabolic mean value formula and the parabolic maximum principle, Energy and uniqueness, regularity estimates.

4 Main references

The basic references for this course is Evans [1]. We would sometimes follow Folland [2], John [3] and some other references for particular materials as well.

References

[1] Evans, L. C. *Partial differential equations*, vol. 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.

[2] Folland, G. B. *Introduction to partial differential equations*, second ed. Princeton University Press, Princeton, NJ, 1995.

[3] John, F. *Partial differential equations*, fourth ed., vol. 1 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1982.