Note on an easy proof of Gaffney for vector fields in dimension 3 for simply connected domains

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Theorem 1 (Vanishing normal part on the boundary) *Let $\Omega \subset \mathbb{R}^3$ be open, bounded, smooth and simply connected. Let $\hat{n}$ denote the exterior unit normal to $\partial\Omega$. Then for any $u \in L^2\left(\Omega;\mathbb{R}^3\right)$ such that $\operatorname{curl} u \in L^2\left(\Omega;\mathbb{R}^3\right)$, $\operatorname{div} u \in L^2\left(\Omega\right)$ and $\hat{n}\cdot u = 0$ on $\partial\Omega$, then $u \in W^{1,2}\left(\Omega;\mathbb{R}^3\right)$ with the estimate*

$$\|\nabla u\|_{L^2}^2 \leq c\left(\|\operatorname{curl} u\|_{L^2}^2 + \|\operatorname{div} u\|_{L^2}^2\right).$$

Proof We divide the proof in three steps.

Step 1 We choose a ball B_R large enough such that $\Omega \subset\subset B_R$. We first find $\phi \in W^{1,2}(B_R \setminus \overline{\Omega})$ such that,

$$\begin{cases} \Delta\phi = 0 & \text{in } B_R \setminus \overline{\Omega}, \\ \dfrac{\partial\phi}{\partial\hat{n}} = \hat{n}\cdot\operatorname{curl} u & \text{on } \partial\Omega, \\ \dfrac{\partial\phi}{\partial\hat{n}} = 0 & \text{on } \partial B_R. \end{cases} \tag{1}$$

Note that the Neumann problem (1) is solvable since

$$\int_{\partial\Omega} \hat{n}\cdot\operatorname{curl} u = \int_{\Omega} \operatorname{div}\left(\operatorname{curl} u\right) = 0.$$

Then, we define $\chi \in L^2(\mathbb{R}^3;\mathbb{R}^3)$ as,

$$\chi = \begin{cases} \operatorname{curl} u & \text{in } \Omega, \\ \nabla\phi & \text{in } B_R \setminus \overline{\Omega}, \\ 0 & \text{if } x \in \mathbb{R}^3 \setminus \overline{B_R}. \end{cases}$$

This implies,

$$\operatorname{div}\chi = 0 \qquad \text{in } \mathbb{R}^3.$$

Indeed, for any $\theta \in C_c^\infty(\mathbb{R}^3)$, we have,

$$\begin{aligned} \int_{\mathbb{R}^3} \langle \chi, \nabla\theta\rangle &= \int_{\Omega} \langle \operatorname{curl} u, \nabla\theta\rangle + \int_{B_R\setminus\overline{\Omega}} \langle \nabla\phi, \nabla\theta\rangle \\ &= \int_{\partial\Omega} (\hat{n}\cdot\operatorname{curl} u)\theta - \int_{\partial\Omega} (\hat{n}\cdot\nabla\phi)\theta + \int_{\partial B_R} (\hat{n}\cdot\nabla\phi)\theta = 0. \end{aligned}$$

Now we find $\psi \in W^{2,2}(\mathbb{R}^3;\mathbb{R}^3)$ such that

$$\Delta\psi = \chi \qquad \text{in } \mathbb{R}^3. \tag{2}$$

Now $\operatorname{div}\chi = 0$ in $\mathbb{R}^3$ implies $\operatorname{div}\psi = 0$ in $\mathbb{R}^3$. Indeed, we have , in $\mathbb{R}^3$,

$$\Delta(\operatorname{div}\psi) = \operatorname{div}(\nabla(\operatorname{div}\psi)) = \operatorname{div}[(\operatorname{curl}\operatorname{curl} + \nabla\operatorname{div})\psi] = \operatorname{div}(\Delta\psi) = \operatorname{div}\chi = 0.$$

Since $\operatorname{div}\psi \in L^2(\mathbb{R}^3)$, this implies $\operatorname{div}\psi = 0$ in $\mathbb{R}^3$. We also have the estimate,

$$\|\psi\|_{W^{2,2}} \le c\|\operatorname{curl} u\|_{L^2}.$$

Step 2 Now we find $\xi \in W^{2,2}(\mathbb{R}^3)$ such that

$$\begin{cases} \Delta\xi = \operatorname{div} u & \text{in } \Omega, \\ \dfrac{\partial\xi}{\partial\hat{n}} = -\hat{n}\cdot\operatorname{curl}\psi & \text{on } \partial\Omega, \end{cases} \tag{3}$$

Note that the Neumann problem (3) is solvable since

$$\int_\Omega \operatorname{div} u = \int_{\partial\Omega} \hat{n}\cdot u = 0 = -\int_\Omega \operatorname{div}(\operatorname{curl}\psi) = -\int_{\partial\Omega} \hat{n}\cdot\operatorname{curl}\psi.$$

We also have the estimate,

$$\|\xi\|_{W^{2,2}} \le c\left(\|\operatorname{div} u\|_{L^2} + \|\operatorname{curl}\psi\|_{W^{\frac{3}{2},2}(\partial\Omega)}\right).$$

Step 3 Now we define

$$h = u - \operatorname{curl}\psi - \nabla\xi.$$

We obtain,

$$\begin{aligned} \operatorname{curl} h &= \operatorname{curl} u - \operatorname{curl}\operatorname{curl}\psi = \operatorname{curl} u - \Delta\psi = 0 && \text{in } \Omega, \\ \operatorname{div} h &= \operatorname{div} u - \operatorname{div}\nabla\xi = \operatorname{div} u - \Delta\xi = 0 && \text{in } \Omega, \\ \hat{n}\cdot h &= \hat{n}\cdot(u - \operatorname{curl}\psi - \nabla\xi) = -\hat{n}\cdot\operatorname{curl}\psi - \frac{\partial\xi}{\partial\hat{n}} = 0 && \text{on } \partial\Omega. \end{aligned}$$

Thus, h is a harmonic field with vanishing normal part on the boundary. Since Ω is simply connected, $h = 0$ and thus

$$u = \operatorname{curl}\psi + \nabla\xi \qquad \text{in } \Omega.$$

Thus, we obtain,

$$\begin{aligned} \|\nabla u\|_{L^2}^2 &\le c\left(\|\nabla(\operatorname{curl}\psi)\|_{L^2}^2 + \|\nabla(\nabla\xi)\|_{L^2}^2\right) \le c\left(\|\psi\|_{W^{2,2}}^2 + \|\xi\|_{W^{2,2}}^2\right) \\ &\le c\left(\|\operatorname{curl} u\|_{L^2}^2 + \|\operatorname{div} u\|_{L^2}^2\right). \end{aligned}$$

This concludes the proof. ■