Nonlinear Stein theorem for Differential Forms

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Outline

Sobolev embedding and Stein theorem

Sobolev and Sobolev-Morrey embedding

$u \in W^{1,p}_{loc}(\mathbb{R}^n), 1 < p < \infty$. (Also true for $p = 1, \infty$). Then

- **Sobolev-Morrey** if $p > n$, then $u \in C^{0,\frac{p-n}{p}}_{loc}(\mathbb{R}^n)$.
- **Critical Sobolev**

$$u \in W^{1,n}_{loc}(\mathbb{R}^n) \not\Rightarrow u \in L^{\infty}_{loc}(\mathbb{R}^n).$$

Example

$u(x) = \log\log\left(1 + \frac{1}{|x|}\right) \in W^{1,n}(B^n_1)$ for $n > 1$, but is unbounded near 0.

Sobolev embedding and Stein theorem

Sobolev and Sobolev-Morrey embedding

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Sharp criterion for continuity

A big gap! $p = n$, not even bounded vs $p = n + \varepsilon$, Hölder continuous.
Is there a borderline space that implies '*just*' continuity?

Lorentz spaces near L^n

Lorentz spaces

$1 < p < \infty$, $1 \leq q \leq \infty$. Interpolation spaces. More refined than $L^p = L^{(p,p)}$.

- $q < \infty$

$$f \in L^{(p,q)} \simeq \int_0^\infty t^q \left|\{x : |f(x)| > t\}\right|^{\frac{q}{p}} \frac{\mathrm{d}t}{t} < \infty.$$

- $q = \infty$ (Weak L^p)

$$f \in L^{(p,\infty)} \simeq \sup_{t>0} \left(t^p \left|\{x : |f(x)| > t\}\right|\right) < \infty.$$

Inclusion of Lorentz spaces near L^n

$$L^q = L^{(q,q)} \subsetneq L^{(n,1)} \subsetneq L^n = L^{(n,n)} \subsetneq L^{(n,\infty)} \quad \text{for any } q > n.$$

Example

$u(x) = \frac{1}{|x| \log^{\beta}\left(\frac{1}{|x|}\right)}$ near zero is $L^{(n,\infty)}$ for $\beta \geq 0$, L^n for $\beta \geq 1$ and $L^{(n,1)}$ for $\beta > 1$.

Campanato spaces

Campanato seminorm

$1 < p < \infty$, $0 \leq \lambda \leq n + p$. $f \in \mathcal{L}^{p,\lambda}$ if $f \in L^p$ with $[f]^p_{\mathcal{L}^{p,\lambda}} < \infty$.

$$[f]^p_{\mathcal{L}^{p,\lambda}(\Omega)} = \sup_{\substack{x \in \Omega \\ 0 < r < \operatorname{diam}(\Omega)}} \frac{1}{r^\lambda} \int_{B_r(x) \cap \Omega} \left| f - (f)_{(B_r(x) \cap \Omega)} \right|^p,$$

where $$(f)_{(B_r(x) \cap \Omega)} := \frac{1}{|B_r(x) \cap \Omega|} \int_{B_r(x) \cap \Omega} f := ⨍_{B_r(x) \cap \Omega} f.$$

- If $n < \lambda \leq n + p$, then $\mathcal{L}^{p,\lambda} \simeq C^{0, \frac{\lambda - n}{p}}$.
- $\mathcal{L}^{p,n} \simeq BMO$ ($p = 1$ is the BMO seminorm). VMO is the closure of C^∞_c functions under BMO seminorm, a strict subspace of BMO.

Example

$\log|x| \in BMO(B_1)$, but not $VMO(B_1)$. $\log^\beta|x| \in VMO(B_1)$ for $0 < \beta < 1$. $\log\log|x| \in VMO(B_1)$. So neither BMO nor VMO is contained in L^∞.

Theorem (Stein 1981, Ann. of Math [11])

$$u \in W^{1,(n,1)}_{loc}(\mathbb{R}^n) \Rightarrow u \in C^0_{loc}(\mathbb{R}^n).$$

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Some other Lorentz-Sobolev embeddings

$$u \in W^{1,(n,\infty)}_{loc}(\mathbb{R}^n) \Rightarrow u \text{ is locally } BMO.$$

$$u \in W^{1,n}_{loc}(\mathbb{R}^n) \Rightarrow u \text{ is locally } VMO.$$

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PDE formulation using CZ estimates

Interpolation spaces $\Rightarrow$ Calderon-Zygmund estimates hold.

$$\Delta u \in L^{(n,1)}_{loc} \Rightarrow \nabla u \in W^{1,(n,1)}_{loc} \qquad \left(\left\|\nabla^2 u\right\|_{L^p} \simeq \|\Delta u\|_{L^p}\right) \qquad \text{(CZ estimates)}$$

$$\Rightarrow \nabla u \text{ is continuous} \qquad \text{(Stein theorem)}$$

Similarly, $\Delta u \in L^{(n,\infty)}_{loc} \Rightarrow \nabla u \in BMO_{loc}$, $\Delta u \in L^n_{loc} \Rightarrow \nabla u \in VMO_{loc}$ and $\Delta u \in L^q_{loc}$ for some $q > n \Rightarrow \nabla u \in C^{0,\beta}_{loc}$ for some $1 < \beta < 1$.

Nonlinear Calderon-Zygmund theory

Nonlinear CZ theory: Scalar case

Uraltseva, **Iwaniec**, **Manfredi**, **DiBenedetto**, **Kilpeläinen**, **Maly**, **Acerbi**, **Fusco**, **Lewis**, **Lindqvist**, **Lieberman**, **Duzaar**, **Mingione**, **Kuusi** and many, many, many others....

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = f$$

For $p \neq 2$, $u \notin C^{\infty}_{loc}$ even for $f = 0$! However, $u \in C^{1,\beta}_{loc}$ for some $0 < \beta < 1$.

- **Long story short:** Gradient estimates still hold for $p > 2$. linear and nonlinear Potential estimates.... which also extends to the general case

$$\operatorname{div} a(\nabla u) = f.$$

- **Kuusi-Mingione (ARMA 2013) [6]** $f \in L^{(n,1)} \Rightarrow \nabla u$ is continuous.

Equations to systems

Nonlinear CZ theory: Vectorial case

$$\operatorname{div} A\left(Du\right) = f \qquad (\text{ Remark: } \operatorname{div} \text{ is row-wise here.})$$

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- Systems are very different! Everywhere regularity is, in general, not true!
- **Uhlenbeck, Acta Math 1977 [12]** (elliptic complexes), **Hamburger, J. Reine. Angew. Math 1992 [5]** (vector-valued differential forms) Everywhere Hölder continuity of Du holds true if $f = 0$ and

$$A(Du) \simeq Dg(Du) \qquad \text{with } g(Du) = g\left(|Du|\right).$$

Uhlenbeck structure, **Quasidiagonal structure**. True for homogeneous p-Laplacian system $\qquad \operatorname{div}\left(|Du|^{p-2}\,Du\right) = 0.$

Nonlinear Stein theorem: Vectorial case

Inhomogeneous systems

$$\operatorname{div}\left(a(x)\,|Du|^{p-2}\,Du\right) = f, \qquad 0 < \gamma \leq a(x) \leq L < \infty.$$

Dini continuity

$a : \Omega \to [\gamma, L]$ is Dini-continuous if there exists a concave, non-decreasing function $\omega : [0, \infty) \to [0, 1]$ (modulus of continuity) with $\omega(0) = 0$ such that for every $x, y \in \Omega$, we have $|a(x) - a(y)| \leq L\omega\left(|x - y|\right)$ and we have

$$\int_0^{\mathrm{diam}(\Omega)} \omega\left(\rho\right) \frac{\mathrm{d}\rho}{\rho} < \infty.$$

Theorem (Kuusi-Mingione, Calc. Var. PDE 2014 [7])

a is Dini continuous , $f \in L^{(n,1)} \Rightarrow Du$ is continuous.

Sharp with respect to the regularity of both the coefficients and the right hand side, already for $p = 2$ and also for L^{∞} bounds for Du.

p-Laplacian for vector-valued form

Analogue for vector-valued form

$u, f : \mathbb{R}^n \to \Lambda^k \mathbb{R}^n \otimes \mathbb{R}^N$, $0 \leq k \leq n-1$. ($k = n-1 \simeq$ Sobolev embedding)

$$d^* \left(a(x) \left| du \right|^{p-2} du \right) = f \qquad (d, d^* \text{ componentwise })$$

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Features -

- $k = 0, N = 1$ — p-Laplacian **equation.**

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- $k = 0, N \geq 2$ — p-Laplacian **system.**

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  u is a solution $\Rightarrow u + \phi$ is a solution for any closed form ϕ, (i.e $d\phi = 0$) $\Rightarrow$
  The kernel is **infinite dimensional**.

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  The kernel is **infinite dimensional**.
  However, elliptic complex structure — $d^* f = 0$ is a necessary condition since $d^* \circ d^* = 0$. Thus, **elliptic modulo the kernel**.

Regularity results

Standing assumptions:

- $n \geq 2$, $N \geq 1$, $1 < p < \infty$, $0 \leq k \leq n-1$,
- $\Omega \subset \mathbb{R}^n$ open, bounded, ($\mathcal{H}^k_T(\Omega) = \{0\}$.)
- $a : \Omega \to [\gamma, L]$ with $0 < \gamma \leq L < \infty$,
- $d^* f = 0$ in Ω in the sense of distributions,
- $u \in W^{1,p}_{loc}\left(\Omega; \Lambda^k \mathbb{R}^n \otimes \mathbb{R}^N\right)$ is a local solution to

$$d^* \left(a(x) \left|du\right|^{p-2} du \right) = f \qquad \text{in } \Omega.$$

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Theorem (Stein theorem for forms (S., Calc. Var. PDE 2019, [10]))

If a is Dini continuous and $f \in L^{(n,1)}_{loc}\left(\Omega; \Lambda^k\mathbb{R}^n \otimes \mathbb{R}^N\right)$, then du is continuous in Ω and ∇u is locally VMO modulo a closed (exact) form.

Campanato estimates for the gradient for $p \geq 2$

$$d^* \left(a(x) \left| du \right|^{p-2} du \right) = d^* F \qquad \text{in } \Omega. \tag{1}$$

Let $p \geq 2$, and let β is the Hölder exponent for $V(dv)$ for the homogeneous constant coefficient system and let $a \in C^{0,\alpha}_{loc}(\Omega)$ and $0 \leq \lambda < \min\{n + 2\alpha, n + 2\beta\}$.

Theorem (Campanato estimate (S., Calc. Var. PDE 2019, [10]))

$F \in \mathcal{L}^{p',\lambda}_{loc} \Rightarrow \nabla u \in \mathcal{L}^{2,\frac{np-2n+2\lambda}{p}}_{loc}$, modulo an closed (exact) form.

This implies, modulo an closed (exact) form, we have

- $f \in L^q_{loc}$ for some $q > n \Rightarrow u \in C^{1,\theta}_{loc}$ for some $0 < \theta < 1$.
- $f \in L^n_{loc} \Rightarrow \nabla u$ is locally VMO.
- $f \in L^{(n,\infty)}_{loc} \Rightarrow \nabla u$ is locally BMO.

This generalizes **DiBenedetto-Manfredi, Amer. J. Math. 1993** [3].

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This generalizes **DiBenedetto-Manfredi, Amer. J. Math. 1993** [3]. See also **Diening-Kaplický-Schwarzacher, Nonlinear Anal. 2012** [4] and **Breit-Cianchi-Diening-Kuusi-Schwarzacher, J. Math. Pures Appl. 2018** [2].

Vector fields in dimension three

$\Omega \subset \mathbb{R}^3$ is open, bounded, contractible, $1 < p < \infty$, $a : \Omega \to [\gamma, L]$ with $0 < \gamma \leq L < \infty$. Let $\operatorname{div} f = 0$ in Ω and $u \in W^{1,p}_{loc}\left(\Omega; \mathbb{R}^3\right)$ is a solution to

$$\operatorname{curl}\left(a(x)\left|\operatorname{curl} u\right|^{p-2}\operatorname{curl} u\right) = f \qquad \text{in } \Omega.$$

Theorem (Stein theorem for vector fields in dimension three)

If a is Dini continuous and $f \in L^{(3,1)}_{loc}\left(\Omega; \mathbb{R}^3\right)$, then $\operatorname{curl} u$ is continuous in Ω.

Theorem (Campanato estimate for vector fields in dimension three)

If $p \geq 2$, $a \in C^{0,\alpha}_{loc}(\Omega)$, then modulo a gradient field, we have

- $f \in L^{(3,\infty)}_{loc}\left(\Omega; \mathbb{R}^3\right) \Rightarrow \nabla u \in BMO_{loc}\left(\Omega; \mathbb{R}^3 \otimes \mathbb{R}^3\right)$,
- $f \in L^{3}_{loc}\left(\Omega; \mathbb{R}^3\right) \Rightarrow \nabla u \in VMO_{loc}\left(\Omega; \mathbb{R}^3 \otimes \mathbb{R}^3\right)$,
- $f \in L^{q}_{loc}\left(\Omega; \mathbb{R}^3\right)$ *for some* $q > 3 \Rightarrow u \in C^{1,\beta}_{loc}\left(\Omega; \mathbb{R}^3\right)$.

Comparison for inhomogeneous system

$u \in W^{1,p}_{loc}\left(\mathbb{R}^n; \mathbb{R}^N\right)$ solves

$$\operatorname{div}\left(a(x)\left|Du\right|^{p-2} Du\right) = f. \tag{2}$$

Pick a point $x_0 \in \mathbb{R}^n$ and we first solve (unique solution exists)

$$\begin{cases} \operatorname{div}\left(a(x)\left|Dw\right|^{p-2} Dw\right) = 0 & \text{in } B_R(x_0) \\ w = u & \text{on } \partial B_R(x_0). \end{cases} \tag{3}$$

Then we solve (unique solution exists)

$$\begin{cases} \operatorname{div}\left(a(x_0)\left|Dv\right|^{p-2} Dv\right) = 0 & \text{in } B_{R/2}(x_0) \\ v = w & \text{on } \partial B_{R/2}(x_0). \end{cases} \tag{4}$$

Comparison for inhomogeneous system

$u - w \in W_0^{1,p}\left(B_R(x_0);\mathbb{R}^N\right)$ + weak formulations of (2) and (3) $\Rightarrow$

$$\int_{B_R(x_0)} a(x)\left\langle |Du|^{p-2} Du - |Dw|^{p-2} Dw; u - w\right\rangle = \int_{B_R(x_0)} \langle f; u - w\rangle . \tag{5}$$

$$\int_{B_R(x_0)} |Du - Dw|^p \leq \text{LHS} \quad \text{by monotonicity } (p \geq 2)$$

$$|\text{RHS}| \leq \left(\int_{B_R(x_0)} |f|^{(p^*)'}\right)^{\frac{1}{(p^*)'}} R \left(\int_{B_R(x_0)} |Du - Dw|^p\right)^{\frac{1}{p}}, \tag{*}$$

by Hölder and Sobolev-Poincaré.

Trouble for forms

Naive analogy can not work.

$$\begin{cases} d^{*}\left(a(x)\left|dw\right|^{p-2}dw\right) = 0 & \text{in } B_R(x_0) \\ w = u & \text{on } \partial B_R(x_0). \end{cases}$$

- No unique solution .
- $\|du - dw\|_{L^p}$ does not control any of the following norms

$$\|\nabla u - \nabla w\|_{L^p}, \|u - w\|_{W^{1,p}}, \|u - w\|_{L^{p^*}}.$$

Gauge fixing

We need to quotient out the kernel and restore ellipticity .

First heuristic idea

The system

$$d^* \left(a(x) \, |du|^{p-2} \, du \right) = f \tag{E1}$$

is (locally) equivalent to

$$d^* \left(a(x) \, |du|^{p-2} \, du \right) = f \qquad \text{and} \qquad d^* u = 0. \tag{E2}$$

Picks out only one, the unique 'nicest' representative from each class $\{ u + \phi : \phi \text{ closed } \} \simeq$ Projection onto the quotient by the kernel.

The space $W^{d,p}_{d^*,T}$

$W^{1,p}_0$ is **not** the correct space. $W^{1,p}_{d^*,T} = W^{d,p}_{d^*,T}$ is!

$$
\begin{aligned}
&W^{d,p}_{d^*,T}\left(B_R(x_0); \Lambda^k\mathbb{R}^n \otimes \mathbb{R}^N\right) \\
&= \left\{u \in L^p : du \in L^p, d^*u = 0 \text{ in } B_R(x_0), \iota^*_{\partial B_R(x_0)} u = 0 \text{ on } \partial B_R(x_0)\right\} \\
&= \overline{C^\infty_c\left(B_R(x_0); \Lambda^k\mathbb{R}^n \otimes \mathbb{R}^N\right) \cap \operatorname{Ker} d^*}^{\|\cdot\|_{W^{d,p}}}
\end{aligned}
$$

Technical gain of gauge fixing

The system (E2) admits an existence theory and a weak formulation in $W^{d,p}_{d^*,T}$.
Look for $u \in W^{d,p}_{d^*,T}\left(B_R(x_0); \Lambda^k\mathbb{R}^n \otimes \mathbb{R}^N\right)$ satisfying

$$\int_{B_R(x_0)} \left\langle a(x)\left|du\right|^{p-2} du; d\phi\right\rangle = \int_{B_R(x_0)} \langle f; \phi\rangle \qquad \text{for all } \phi \in W^{d,p}_{d^*,T}.$$

Comparison systems

Second idea

For comparison, use the systems

$$\begin{cases} d^{*}\left(a(x)\left|dw\right|^{p-2}dw\right)=0 & \text{in } B_R(x_0)\\ d^{*}w=d^{*}u & \text{in } B_R(x_0)\\ \iota^{*}_{\partial B_R(x_0)}w=\iota^{*}_{\partial B_R(x_0)}u & \text{on } \partial B_R(x_0). \end{cases}$$

and

$$\begin{cases} d^{*}\left(a(x_0)\left|dv\right|^{p-2}dv\right)=0 & \text{in } B_{R/2}(x_0)\\ d^{*}v=d^{*}w & \text{in } B_{R/2}(x_0)\\ \iota^{*}_{\partial B_R(x_0)}v=\iota^{*}_{\partial B_{R/2}(x_0)}w & \text{on } \partial B_{R/2}(x_0). \end{cases}$$

Existence and uniqueness for comparison systems

- Existence and uniqueness of solutions. (modulo cohomology).
- Solutions are unique minimizers for

$$\text{Minimize} \qquad m = \inf \left\{ \frac{1}{p} \int_{B_R} a(x) \left| du \right|^p : u \in u_0 + W^{d,p}_{d^*,T} \right\}.$$

Clearly, the weak formulations in $W^{d,p}_{d^*,T}$ is valid.
Minimization on $u_0 + W^{1,p}_0$ is possible, intimately related.
Bandyopadhyay-Dacorogna-S., JEMS 2015 [1]($N = 1$);
S., Adv. Calc. Var 2019 [9] ($N \geq 2$).
Allows one to work with the naive analogy for the case $f = d^* F$ and obtain analogues of **Diening-Kaplický-Schwarzacher, Nonlinear Anal. 2012** [4] and **Breit-Cianchi-Diening-Kuusi-Schwarzacher, J. Math. Pures Appl. 2018** [2] by deriving estimate for $A(du) := \left| du \right|^{p-2} du$.

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$W^{d,p}_{d^*,T}$ again

Gaffney inequality

L^p and Campanato estimates for the linear elliptic system

$$\begin{aligned} du = f \qquad \text{and} \qquad d^* u = g \qquad & \text{in } B_R, \\ \iota^*_{\partial B_R} u = \iota^*_{\partial B_R} u_0 \qquad & \text{on } \partial B_R. \end{aligned}$$

$$\|\nabla u\|_{L^p(B_R)} \leq C \left(\|f\|_{L^p(B_R)} + \|g\|_{L^p(B_R)} + \|u_0\|_{W^{1-\frac{1}{p},p}(\partial B_R)} \right)$$

B_R has trivial cohomology. $W^{d,p}_{d^*,T} = W^{1,p}_{d^*,T}$.

Poincaré-Sobolev inequality

If $u \in W^{d,p}_{d^*,T}\left(B_R; \Lambda^k \mathbb{R}^n \otimes \mathbb{R}^N\right)$, $1 < p < n$ and $p^* = \frac{np}{n-p}$, then

$$\left(\fint_{B_R} |u|^{p^*} \right)^{\frac{1}{p^*}} \leq cR \left(\fint_{B_R} |du|^p \right)^{\frac{1}{p}}.$$

Campanato estimate for $p \geq 2$

- Basically linear estimates for the nonlinear quantity

$$V(du) = |du|^{\frac{p-2}{2}} du, \qquad \text{Also possible using } A(du) = |du|^{p-2} du.$$

- Comparison estimates

$$\int_{B_R} |V(du) - V(dw)|^2 \leq c \int_{B_R} \left| F - (F)_{B_R} \right|^{\frac{p}{p-1}}.$$

$$\int_{B_R} |V(dv) - V(dw)|^2 \leq c \left[\omega(R)\right]^2 \int_{B_R} |V(dw)|^2.$$

$$\int_{B_\rho} \left| V(dv) - (V(dv))_{B_\rho} \right|^2 \leq c \left(\frac{\rho}{R}\right)^{n+2\beta_2} \int_{B_R} \left| V(dv) - (V(dv))_{B_R} \right|^2.$$

- From $V(du)$ to du

$$V(du) \in \mathcal{L}^{2,\lambda} \Rightarrow du \in \mathcal{L}^{p,\lambda} \Rightarrow du \in \mathcal{L}^{2,\frac{np-2n+2\lambda}{p}}$$

- From du to ∇u : $du \in \mathcal{L}^{2,\frac{np-2n+2\lambda}{p}}$ and $d^*u = 0 \Rightarrow \nabla u \in \mathcal{L}^{2,\frac{np-2n+2\lambda}{p}}$.

Homogeneous system with Dini coefficients for $p \geq 2$

To prove the Stein theorem result, as an intermediate step we need to prove the theorem for $p \geq 2$ and $f = 0$, i.e. for the system

$$d^* \left(a(x) \left| dw \right|^{p-2} dw \right) = 0 \qquad \text{in } \Omega.$$

Basic strategy

- First prove for any ball $B_R \subset\subset \Omega$, the L^∞ estimate,

$$\sup_{B_{R/2}} \left| dw \right| \leq c ⨍_{B_R} \left| dw \right|.$$

- Then using the L^∞ bound to show that the continuous maps

$$\alpha_i(x) := ⨍_{B_{R_i}(x)} dw$$

  converge uniformly as $i \to \infty$ on any compact subset $K \subset \Omega$. Thus, the limit is a continuous map which agrees a.e. with dw. Hence dw is continuous.

Pointwise estimate for homogeneous system with Dini coefficients for $p \geq 2$

- Comparison estimates in shrinking ball $B_i := B_{R_i}$ with $R_i := \sigma^i R$.

$$\fint_{B_i} |V(dv_i) - V(dw)|^2 \leq c_4 \left[\omega\left(R_i\right)\right]^2 \fint_{B_i} |V(dw)|^2 .$$

- Set $\lambda^{\frac{p}{2}} := H_1 \left(\fint_{B_R} |V(dw)|^2 \right)^{\frac{1}{2}}$ and choose H_1 large, R and σ small.
- Excess decay: Let $E_2\left(V(dw), B_i\right) := \left(\fint_{B_i} \left|V(dw) - (V(dw))_{B_i}\right|^2 \right)^{\frac{1}{2}}$

$$\fint_{B_i} |V(dw)|^2 \leq \lambda^p \Rightarrow E_2\left(V(dw), B_{i+1}\right) \leq \frac{1}{4} E_2\left(V(dw), B_i\right) + c\lambda^{\frac{p}{2}}\omega(R_i).$$

- Prove by induction that $\left|(V(dw))_{B_i}\right| + E_2\left(V(dw), B_i\right) \leq \lambda^{\frac{p}{2}}$ for all i.
- $|dw(x)| \leq \liminf_{i\to\infty} \left|(dw)_{B_i}\right| \leq \left(\fint_{B_i} |V(dw)| \right)^{\frac{2}{p}} \leq \lambda$, for any Lebesgue point x.

Stein theorem

Basic strategy

- First prove for any ball $B_R \subset\subset \Omega$ and every Lebesgue point x of du, the pointwise estimate,

$$|du(x)| \leq c \left(\fint_{B_R} |du|^s \right)^{\frac{1}{s}} + \|f\|_{L^{(n,1)}}^{\frac{1}{p-1}},$$

where $s = p^{'}$ if $p > 2$ and $s = p$ if $1 < p < 2$.

- Then using the L^∞ bound to show that the continuous maps

$$\alpha_i(x) := \fint_{B_{R_i}(x)} du$$

converge uniformly as $i \to \infty$ on any compact subset $K \subset \Omega$. Thus, the limit is a continuous map which agrees a.e. with du. Hence du is continuous.

Considerably more involved estimates. However, with our comparison systems and our Poincaré-Sobolev inequality, becomes exactly analogous to the $k = 0$ case.

Next...

Moral of the story so far

Estimates for ∇u for the inhomogeneous p-Laplacian type systems
$\rightsquigarrow$ Analogous estimates for du for the inhomogeneous systems for forms .

How far is this the general picture? only for du or valid for the full gradient?

Possible extensions

- Campanato estimate for the gradient for $1 < p < 2$?

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  Estimate for $A(du)$ is possible by following the work by Diening and collaborators in [4] and [2] with the existence issue fixed by [1] and [9]. But in this range of p, those estimates does not imply estimates for du or ∇u.

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- linear and nonlinear potential estimates? (Typically Riesz potential bounds for the gradient and Wolff potential bounds for the form). Work in progress. Note that Morrey estimates i.e. a Nonlinear Adams theorem would be implied by potential estimates for the gradient.

Gradient estimates and nonlinear Sobolev embedding

An **interesting** (and probably difficult) question:
For u coclosed ($d^*u = 0$), if

$$d^*\left(|du|^{p-2}du\right) \in L^{(n,1)}_{loc} \qquad \Rightarrow? \quad u \in C^1_{loc}?\ u \in C^{0,1}_{loc}?$$

Positive answer to the first question yields an improved pointwise nonlinear Stein theorem for scalar functions!

$$\nabla\left(|v|^{p-2}v\right) \in L^{(n,1)}_{loc} \Rightarrow v \text{ is locally the Laplacian of a } C^2(C^{1,1}) \text{ function ?}$$

Note that not every continuous function is the Laplacian of a C^2 function. For $k = n-1$, the system is

$$\nabla\left(|\operatorname{div} u|^{p-2}\operatorname{div} u\right) \in L^{(n,1)}_{loc}$$

Write

$$v = \Delta\psi = \operatorname{div}(\nabla\psi) \quad \text{and} \quad u = \nabla\psi.$$

Then $u \in C^1_{loc}(C^{0,1}_{loc}) \Rightarrow \psi \in C^2_{loc}(C^{1,1}_{loc})$.

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Then $u \in C^1_{loc}(C^{0,1}_{loc}) \Rightarrow \psi \in C^2_{loc}(C^{1,1}_{loc})$. Note that $p = 2$ case is implied by Stein theorem and CZ estimates.

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Thank you

Questions?