Regularity for systems with prescribed tangential or normal part

Swarnendu Sil

École Polytechnique Fédérale de Lausanne
Lausanne, Switzerland

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TIFR-CAM
Bangalore, India

Maxwell's equations in time-harmonic form

Time harmonic Maxwell's equations in three dimensions

$$\begin{cases} \operatorname{curl} H = i\omega\varepsilon E + J_e & \text{in } \Omega, \\ \operatorname{curl} E = -i\omega\mu H + J_m & \text{in } \Omega, \\ \nu \times E = \nu \times E_0 & \text{on } \partial\Omega. \end{cases}$$

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$$\delta du + d\delta u = f \qquad \text{in } \Omega,$$

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Other proof

Agmon-Douglis-Nirenberg or Lopatinskij-Shapiro condition.

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