Distinguished Varieties

Tirthankar Bhattacharyya

Department of Mathematics
Indian Institute of Science
Bangalore

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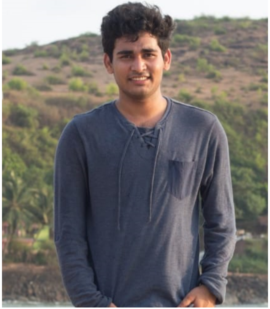

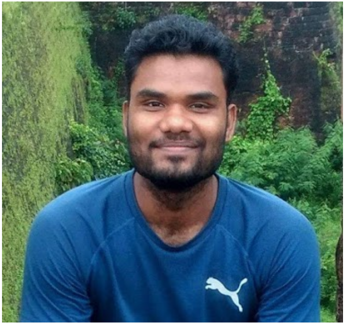

Joint work with Poornendu Kumar who is a PhD student at IISc Bangalore and Haripada Sau who is an INSPIRE faculty at TIFR Bangalore.

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- $\nu(A)$: the numerical radius of a square matrix A;
- $\sigma_T(T_1, T_2, \ldots, T_d)$: The Taylor joint spectrum of a commuting $d - tuple$ of operators $(T_1, T_2, \ldots, T_d)$.

Reference

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Distinguished Varieties Through the Berger–Coburn–Lebow Theorem,
T. Bhattacharyya, P. Kumar, H. Sau,
arXiv:2001.01410

Definition (Algebraic Variety)

A subset $\mathcal{W} \subset \mathbb{C}^d$ is called an *(algebraic) variety* if

$$\mathcal{W} = \{(z_1, z_2, ..., z_d) \in \mathbb{C}^d : \xi_\alpha(z_1, z_2, ..., z_d) = 0 \text{ for all } \alpha \in \Lambda\}$$

where Λ is an index set and ξ_α are in $\mathcal{C}[z_1, z_2, ..., z_d]$.

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Definition (Distinguished Boundary)

The distinguished boundary $b\mathbb{D}^2$ is defined to be the smallest closed subset C of $\overline{\mathbb{D}^2}$ such that every function in $\mathcal{A}(\mathbb{D}^2)$, attains its maximum modulus on C. It turns out that $b\mathbb{D}^2 = \mathbb{T}^2$.

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Definition (Distinguished variety)

A variety W is called a distinguished variety with respect to $\mathbb{D}^2$ if

$$W \cap \mathbb{D}^2 \neq \emptyset \text{ and } W \cap \partial\mathbb{D}^2 = W \cap b\mathbb{D}^2.$$

Matrix rational inner functions

A matrix valued holomorphic function Ψ on $\mathbb{D}$ is said to be *rational inner* if there is a block unitary matrix $\mathcal{U} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ such that $\Psi(z) = A + zB(I - zD)^{-1}C$.

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Obviously, there is a matrix polynomial F and scalar polynomial q such that $\Psi(z) = F(z)/q(z)$ and there is no factor of q that divides every entry of F. The poles of Ψ, i.e., the zeros of q, are away from $\overline{\mathbb{D}}$.

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If $\mathcal{W}_\Psi$ is the algebraic variety

$$\mathcal{W}_\Psi := \{(z_1, z_2) \in \mathbb{C}^2 : \det(F(z_1) - z_2 q(z_1) I) = 0\},$$

then $\mathbb{D}^2 \cap \mathcal{W}_\Psi = \{(z_1, z_2) \in \mathbb{D}^2 : \det(\Psi(z_1) - z_2 I) = 0\}$.

The Agler-McCarthy-Knese Theorem

Theorem

Let Ψ and $\mathcal{W}_\Psi$ be as above. Then the following are equivalent:

(i) $\mathcal{W}_\Psi$ is a distinguished variety with respect to $\mathbb{D}^2$;

(ii) $\nu(\Psi(z)) < 1$ for all z in $\mathbb{D}$;

Conversely, if $\mathcal{W}$ is any distinguished variety with respect to $\mathbb{D}^2$, then there is a matrix-valued rational inner function Ψ on $\mathbb{D}$ such that

$$\mathcal{W} \cap \mathbb{D}^2 = \mathcal{W}_\Psi \cap \mathbb{D}^2.$$

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The proof of the converse part consists of manufacturing a pair of commuting pure isometries on a Hilbert space of functions supported on $\mathcal{W} \cap \mathbb{T}^2$. This pair is then modelled as (M_z, M_Ψ) in the Sz.-Nagy–Foias style.

The polynomial ξ_Ψ

Let α be a zero of q. If there is a β such that $(\alpha, \beta) \in \mathcal{W}_\Psi$, then $\det F(\alpha) = 0$. This means that (α, z_2) is a zero of the polynomial $\det(F(z_1) - z_2 q(z_1) I)$ for every z_2. Thus, there is an $m_\alpha \geq 1$ such that $\det(F(z_1) - z_2 q(z_1) I)$ is divisible by $(z_1 - \alpha)^{m_\alpha}$. Take the largest such m_α for every α that is a zero of q. Then, there is a polynomial ξ_Ψ such that

$$\det(F(z_1) - z_2 q(z_1) I) = \prod_{\alpha \in Z(q)} (z_1 - \alpha)^{m_\alpha} \xi_\Psi(z_1, z_2) \qquad (1)$$

with the understanding that m_α could be 0 for some α (precisely those α for which there is no β satisfying $(\alpha, \beta) \in \mathcal{W}_\Psi$).

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After an example, we shall restate the Agler–McCarthy–Knese theorem in a way that allows a description of the whole variety instead of just its portion in $\mathbb{D}^2$, under a natural condition. We use the notations $\mathbb{E}$ for the complement of $\overline{\mathbb{D}}$ in $\mathbb{C}$.

An example

To illustrate the idea above, consider the 2×2 rational inner function

$$\Psi(z) = \begin{pmatrix} \frac{z-a}{1-\bar{a}z} & 0 \\ 0 & \frac{z-b}{1-\bar{b}z} \end{pmatrix}$$

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Then,

$$F(z) = \begin{pmatrix} (1-\bar{b}z)(z-a) & 0 \\ 0 & (1-\bar{a}z)(z-b) \end{pmatrix},$$

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$q(z) = (1-\bar{a}z)(1-\bar{b}z)$ and

$$\det(F(z_1) - z_2 q(z_1) I) =$$
$$(1-\bar{a}z_1)(1-\bar{b}z_1)\big[(z_1-a) - z_2(1-\bar{a}z_1)\big]\big[(z_1-b) - z_2(1-\bar{b}z_1)\big].$$

An improved Agler-McCarthy-Knese Theorem

Theorem

Let Ψ and $\mathcal{W}_\Psi$ be as above. Then the following are equivalent:

(i) $\mathcal{W}_\Psi$ is a distinguished variety with respect to $\mathbb{D}^2$;

(ii) $\nu(\Psi(z)) < 1$ for all z in $\mathbb{D}$;

(iii) $Z(\xi_\Psi) \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$;

(iv) $Z(\xi_\Psi)$ is a distinguished variety.

Conversely, if $\mathcal{W}$ is any distinguished variety with respect to $\mathbb{D}^2$, then there is a matrix-valued rational inner function Ψ on $\mathbb{D}$ such that

$$\mathcal{W} \cap \mathbb{D}^2 = \mathcal{W}_\Psi \cap \mathbb{D}^2 = Z(\xi_\Psi) \cap \mathbb{D}^2.$$

Moreover, $\mathcal{W} = Z(\xi_\Psi)$ if and only if both $\mathcal{W}$ and $Z(\xi_\Psi)$ are contained in $\mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$.

The Berger Coburn Lebow Theorem

Consider a projection P and a unitary U on a Hilbert space $\mathcal{F}$. Define two $\mathcal{B}(\mathcal{F})$-valued functions

$$\Phi(z) = P^{\perp}U + zPU \text{ and } \Psi(z) = U^*P + zU^*P^{\perp}.$$

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Easy to see that these are commuting isometries.

If we consider any Hilbert space $\mathcal{L}$ and any commuting unitary operator W_1 and W_2 on $\mathcal{L}$, then $(M_\Phi \oplus W_1, M_\Psi \oplus W_2)$ on $\mathcal{H}^2(\mathcal{F}) \oplus \mathcal{L}$ is also a pair of commuting isometries.

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The Berger–Coburn–Lebow theorem says that any pair of commuting isometries is of this form.

Theorem (Berger–Coburn–Lebow)

Let (V_1, V_2) be a commuting pair of isometries acting on $\mathcal{H}$. Then the space $\mathcal{H}$ breaks into a direct sum of reducing subspaces $\mathcal{H} = \mathcal{H}_u \oplus \mathcal{H}_u^\perp$ such that

1. $V_1|_{\mathcal{H}_u}$ and $V_2|_{\mathcal{H}_u}$ are unitary operators,
2. there exist a Hilbert space $\mathcal{F}$, a unitary U and a projection P on $\mathcal{F}$ such that the pair $(V_1|_{\mathcal{H}_u^\perp}, V_2|_{\mathcal{H}_u^\perp})$ is jointly unitarily equivalent to the commuting pair of multiplication operators (M_Φ, M_Ψ) on the vector valued Hardy space $\mathcal{H}^2(\mathcal{F})$ where Φ and Ψ are the operator-valued functions

$$\Phi(z) = P^\perp U + zPU \text{ and } \Psi(z) = U^* P + zU^* P^\perp.$$

Moreover, if $V = V_1 V_2$, then

$$\mathcal{H}_u = \{x \in \mathcal{H} \mid x = V^n y_n \text{ for all } n > 0\}.$$

Definitions

Definition

A triple $\chi = (\mathcal{F}, P, U)$ where $\mathcal{F}$ is a Hilbert space, P is a projection and U is a unitary operator on $\mathcal{F}$ will be called a *model triple*. If, moreover, $\mathcal{F}$ is finite, then we call it a *finite model triple*.

Let $\Phi(z) = P^{\perp}U + zPU$ and $\Psi(z) = U^*P + zU^*P^{\perp}$.

Definition

For an orthogonal projection P and a unitary U acting on a finite dimensional Hilbert space $\mathcal{F}$, consider the set

$$\mathcal{W}_{P,U} := \{(z_1, z_2) \in \mathbb{C}^2 : (z_1, z_2) \in \sigma_T(P^{\perp}U + z_1z_2PU, U^*P + z_1z_2U^*P^{\perp})\}.$$

Definition

A variety $\mathcal{W}$ is said to be *symmetric* if for any $z_1 \neq 0$ and $z_2 \neq 0$, we have

$$(z_1, z_2) \in \mathcal{W} \text{ if and only if } \left(\frac{1}{\overline{z_1}}, \frac{1}{\overline{z_2}}\right) \in \mathcal{W}. \quad (2)$$

Theorem (A new description of a distinguished variety)

Given a finite model triple $\chi = (\mathcal{F}, P, U)$, the set $\mathcal{W}_{P,U}$ is a symmetric algebraic variety in $\mathbb{C}^2$ for which the following are equivalent:

(i) *$\mathcal{W}_{P,U}$ is a distinguished variety with respect to $\mathbb{D}^2$;*

(ii) *For all z in the open unit disc $\mathbb{D}$,*

$$\nu(U^*(P + zP^{\perp})) < 1 \text{ and } \nu((P^{\perp} + zP)U) < 1; \tag{3}$$

(iii) *$\mathcal{W}_{P,U} \subset \mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$.*

Moreover, $\mathcal{W}_{P,U}$ can be written as

$$\mathcal{W}_{P,U} = \bigcup_{z \in \mathbb{C}} \sigma_T(P^{\perp}U + zPU, U^*P + zU^*P^{\perp}). \tag{4}$$

Theorem continued...

Theorem

Conversely, if $\mathcal{W}$ is any distinguished variety with respect to $\mathbb{D}^2$, then there exist an orthogonal projection P and a unitary U acting on a finite dimensional Hilbert space $\mathcal{F}$ such that

$$\mathcal{W} \cap \mathbb{D}^2 = \mathcal{W}_{P,U} \cap \mathbb{D}^2.$$

Moreover, $\mathcal{W} = \mathcal{W}_{P,U}$ if and only if both $\mathcal{W}$ and $\mathcal{W}_{P,U}$ are contained in $\mathbb{D}^2 \cup \mathbb{T}^2 \cup \mathbb{E}^2$.

Some childlike examples

Let $\mathcal{H}$ be a Hilbert space. Consider $\mathcal{H} \oplus \mathcal{H}$ and let P be the projection onto $\mathcal{H} \oplus 0$. Consider the non-trivial unitary obtained from the $\mathbb{Z}_2$-action on $\mathcal{H} \oplus \mathcal{H}$, i.e., $U(x \oplus y) = y \oplus x$.

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Then, the numerical radius condition is satisfied. The resulting distinguished variety is $\{(z, z) : z \in \mathbb{C}\}$.

The unitary $\imath U$, where U is the unitary above also satisfies the numerical radius condition and the resulting distinguished variety is $\{(z, -z) : z \in \mathbb{C}\}$.

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$\{(z, -z) : z \in \mathbb{C}\}$.

Now, we go for a somewhat non-trivial variety.

Neil parabola

A model triple $(\mathcal{F}, P, U)$ for the Neil parabola $\{(z_1, z_2) \in \mathbb{D}^2 : z_1^3 = z_2^2\}$ is given by

$$\mathcal{F} = \mathbb{C}^5, \quad P = P_{\mathbb{C}^2 \oplus \{0_{\mathbb{C}^3}\}} \quad \text{and} \quad U = E_\sigma,$$

where E_σ is the permutation matrix induced by the permutation $\sigma = (13452)$ in S_5, i.e., .

$$U = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Neil parabola continued...

Indeed, a simple matrix computation gives us the following

$$\Phi(z_1z_2) = \begin{bmatrix} 0 & z_1z_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z_1z_2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \ \Psi(z_1z_2) = \begin{bmatrix} 0 & 0 & z_1z_2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & z_1z_2 & 0 \\ 0 & 0 & 0 & 0 & z_1z_2 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

A not very lengthy calculation yields that the set

$$\Omega_{P,U} = \{(z_1, z_2) \in \mathbb{D}^2 : (z_1, z_2) \in \sigma_T(P^\perp U + z_1z_2PU, U^*P + z_1z_2U^*P^\perp)\}$$

is the same as the Neil parabola.

Neil parabola continued...

More generally, one can check by a somewhat tedious computation that a model triple for the distinguished variety

$$\mathcal{V}_{n,m} = \{(z_1, z_2) \in \mathbb{D}^2 : z_1^n = z_2^m\}; \quad n, m \geq 1 \tag{5}$$

is given by

$$\mathcal{F} = \mathbb{C}^{m+n}, \quad P = P_{\mathbb{C}^m \oplus \{0_{\mathbb{C}^n}\}}, \quad U^* = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where B is the $m \times n$ matrix with 1 at the $(1,1)$ entry and zero elsewhere, C is the $n \times m$ matrix with 1 at the (n, m) entry and zero elsewhere, D is the $n \times n$ upper triangular matrix with 1 in the super diagonal entries and zero elsewhere, and A is the $m \times m$ matrix given as

$$A = \begin{bmatrix} 0 & 0 \\ I_{m-1} & 0 \end{bmatrix}.$$

The canonicity

What is special about the linear pencils $P^{\perp}U + zPU$ and $U^*P + zU^*P^{\perp}$? One could start with any two matrix-valued rational inner functions Φ and Ψ on $\mathbb{D}$ such that

(i) the maps $z \mapsto \nu(\Phi(z))$ and $z \mapsto \nu(\Psi(z))$ are non-constant on $\mathbb{D}$;

(ii) for each $z \in \mathbb{D}$, the pair of matrices $(\Phi(z), \Psi(z))$ is commuting; and

(iii) $\Phi(z)\Psi(z) = z$ for all $z \in \mathbb{D}$.

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(iii) $\Phi(z)\Psi(z) = z$ for all $z \in \mathbb{D}$.

Then $\mathcal{W}_{\Phi,\Psi} := \{(z_1, z_2) \in \mathbb{C}^2 : (z_1, z_2) \in \sigma_T(\Phi(z_1 z_2), \Psi(z_1 z_2))\}$ is a distinguished variety with respect to $\mathbb{C}^2$. The proof is along the same line as the proof of the forward direction of Theorem 4.

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Then $\mathcal{W}_{\Phi,\Psi} := \{(z_1, z_2) \in \mathbb{C}^2 : (z_1, z_2) \in \sigma_T(\Phi(z_1 z_2), \Psi(z_1 z_2))\}$ is a distinguished variety with respect to $\mathbb{C}^2$. The proof is along the same line as the proof of the forward direction of Theorem 4.

It is a consequence of the Berger–Coburn–Lebow Theorem that any such pair of functions is jointly unitarily equivalent to $(P^{\perp}U + zPU, U^*P + zU^*P^{\perp})$ for some model triple $(\mathcal{F}, P, U)$.

Definition (Symmetrized bidisc)

The symmetrized bidisc $\mathbb{G}$, closed symmetrized bidisc Γ and distinguished boundary of the symmetrized bidisc $b\Gamma$ are defined in the following way:

$$\mathbb{G} = \{(z_1 + z_2, z_1 z_2) : |z_1| < 1, |z_2| < 1\}$$

$$\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1| \leq 1, |z_2| \leq 1\}$$

$$b\Gamma = \{(z_1 + z_2, z_1 z_2) : |z_1| = 1, |z_2| = 1\}$$

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Definition (Distinguished variety in $\mathbb{G}$)

A variety V in $\mathbb{G}$ is just $W \cap \mathbb{G}$ for an algebraic variety W in $\mathbb{C}^2$. It is called a distinguished variety if

$$W \cap \partial\mathbb{G} = W \cap b\mathbb{G}$$

Theorem (Pal-Shalit)

Let F be a square matrix with $w(F) < 1$. Let W_F be the subset of $\mathbb{G}$ defined by

$$W_F = \{(s,p) \in \mathbb{G} : det(F + pF^* - sI) = 0\}.$$

Then W_F is a distinguished variety. Conversely, every distinguished variety in $\mathbb{G}$ has the form $\{(s,p) \in \mathbb{G} : det(F + pF^* - sI) = 0\}$, for some matrix F with $w(F) \leq 1$.

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Reference; S. Pal and O. M. Shalit, Spectral sets and distinguished varieties in the symmetrized bidisc, J. Funct. Anal. 266 (2014), 5779-5800

A refinement of Pal-Shalit Theorem

Theorem

Let $F = PU + U^*P^\perp$ for some unitary U and projection P. Suppose $w(F) < 1$. Let V_F be the subset of $\mathbb{G}$ defined by

$$V_F = \{(s,p) \in \mathbb{G} : det(F + pF^* - sI) = 0\}.$$

Then V_F is a distinguished variety. Conversely, given a distinguished variety V in $\mathbb{G}$, there is a unitary U and projection P such that V has the form $\{(s,p) \in \mathbb{G} : det(F + pF^* - sI) = 0\}$, where $F = PU + U^*P^\perp$.

A refinement of Pal-Shalit Theorem

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This theorem is an application of our main theorem and the spectral mapping theorem.

Why is this result a refinement? It is a refinement because while every operator of the form $PU + U^*P^\perp$ has numerical radius no larger than 1, the converse is not true, i.e., there are F with $w(F) \leq 1$ but F can not written in the form $PU + U^*P^\perp$.

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For a non-real complex number α in the open unit disc $\mathbb{D}$ and a Hilbert space $\mathcal{H}$ of any dimension, it is straightforward to see that αI, which has numerical radius less than 1, cannot be written as $PU + U^*P^\perp$ for any projection P and any unitary U coming from $\mathcal{B}(\mathcal{H})$. In case α is real, the dimension needs to be even to write $\alpha I = PU + U^*P^\perp$. The lemma in the next slide gives a larger class of examples.

Lemma

Let $A \in M_2(\mathbb{C})$ *be such that the two eigenvalues* λ_1 *and* λ_2 *satisfy*

$$|\lambda_1| \neq |\lambda_2|. \tag{6}$$

Then A *can not be written as* $PU + U^* P^\perp$ *for any projection* P *and unitary* U*.*

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A realization formula

Theorem (A new realization formula)

A model triple $\chi = (\mathcal{F}, P, U)$ *gives rise to a contractive analytic function* $\Psi_\chi : \mathbb{D} \to \mathcal{B}(\operatorname{Ran} P)$ *defined by*

$$\Psi_\chi(z) = P(I_{\mathcal{F}} - zU^* P^\perp)^{-1} U^* P|_{\operatorname{Ran} P}. \tag{7}$$

Conversely, if $\mathcal{H}$ *is a Hilbert space and* $\Psi : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ *is a contractive analytic function, then there exists a model triple* $\chi = (\mathcal{F}, P, U)$ *such that* $\mathcal{F} \supset \mathcal{H}$, P *is the orthogonal projection of* $\mathcal{F}$ *onto* $\mathcal{H}$ *and* $\Psi = \Psi_\chi$. *Moreover, when* $\mathcal{H}$ *is finite dimensional and* Ψ *is rational inner, then the model triple above can be chosen to be finite.*

AM description vs. BCL description

If $\chi = (\mathcal{F}, P, U)$ is a finite model triple, then Ψ_χ is clearly a rational matrix-valued inner function. Our new description of a distinguished variety of the bidisc is tied up with that of Agler and McCarthy as follows.

Theorem (The passage between two descriptions)

Let $\mathcal{V}$ be a distinguished variety. If $\chi = (\mathcal{F}, P, U)$ is a finite model triple corresponding to $\mathcal{V}$ (i.e., $\mathcal{V} = \Omega_{P,U}$), then $\mathcal{V} = \Omega_{\Psi_\chi}$.
Conversely, let Ψ be a rational matrix-valued inner function which satisfies $\mathcal{V} = \Omega_\Psi$. Let χ be a finite model triple associated to Ψ obtained from the theorem in the previous slide. Then $\mathcal{V} = \Omega_{P,U}$.

A canonical model triple

Two contractive analytic functions $(\Psi_1, \mathcal{E}_1)$ and $(\Psi_2, \mathcal{E}_2)$ are said to be *unitarily equivalent* if there is a unitary operator $\tau : \mathcal{E}_1 \to \mathcal{E}_2$ such that $\tau\Psi_1(z) = \Psi_2(z)\tau$ for all $z \in \mathbb{D}$.

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Consider the two categories

$$\mathfrak{B} := \{(\mathcal{F}, P, U) : \ P \text{ is a projection and } U \text{ is a unitary on } \mathcal{F}\}$$

with the morphisms between two elements $(\mathcal{F}_1, P_1, U_1)$ and $(\mathcal{F}_2, P_2, U_2)$ defined as a linear operator $\tau : \mathcal{F}_1 \to \mathcal{F}_2$ that satisfies

$$\tau(P_1, U_1) = (P_2, U_2)\tau; \tag{8}$$

and

$$\mathfrak{C} = \{(\Psi, \mathcal{E}) : \Psi : \mathbb{D} \to \mathcal{B}(\mathcal{E}) \text{ is analytic and contractive}\}$$

with the morphisms between two elements $(\Psi_1, \mathcal{E}_1)$ and $(\Psi_2, \mathcal{E}_2)$ defined as a linear operator $\tau : \mathcal{E}_1 \to \mathcal{E}_2$ that satisfies

$$\tau\Psi_1(z) = \Psi_2(z)\tau \text{ for all } z \in \mathbb{D}. \tag{9}$$

A canonical model triple

Corresponding to an object $\chi = (\mathcal{F}, P, U)$ in $\mathfrak{B}$, we have an object $\Psi_{P,U} : \mathbb{D} \to \mathcal{B}(\text{Ran}\, P)$ given by is the function

$$\Psi_{P,U}(z) = P(I_{\mathcal{F}} - zU^* P^{\perp})^{-1} U^* P|_{\text{Ran}\, P}.$$

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Let $\chi_1 = (\mathcal{F}_1, P_1, U_1)$ and $\chi_2 = (\mathcal{F}_2, P_2, U_2)$ be two objects in $\mathfrak{B}$ and let τ be a morphism between them. It is easy to see that τ takes the following operator matrix form

$$\tau = \begin{bmatrix} \tau_* & 0 \\ 0 & \tau_{**} \end{bmatrix} : \begin{bmatrix} \text{Ran } P_1 \\ \text{Ran } P_1^{\perp} \end{bmatrix} \to \begin{bmatrix} \text{Ran } P_2 \\ \text{Ran } P_2^{\perp} \end{bmatrix}.$$

The linear transformation $\tau_* : \text{Ran } P_1 \to \text{Ran } P_2$ induced by τ is easily seen to have the property

$$\tau_* \Psi_{P_1,U_1}(z) = \Psi_{P_2,U_2}(z)\tau_* \text{ for all } z \in \mathbb{D}.$$

Thus τ_* is a morphism between the objects $(\Psi_{P_1,U_1}, \text{Ran } P_1)$ and $(\Psi_{P_2,U_2}, \text{Ran } P_2)$. These morphisms will be referred to as the *induced* morphisms.

A canonical model triple

Theorem

The map $\mathfrak{f} : \mathfrak{B} \to \mathfrak{C}$ *defined as*

$$\mathfrak{f} : ((\mathcal{F}, P, U), \tau) \mapsto (\Psi_{P,U}, \tau_*)$$

has the functorial properties, i.e.,

(i) *if* $\iota : (\mathcal{F}, P, U) \to (\mathcal{F}, P, U)$ *is the identity morphism, then the induced morphism* $\iota_* : (\Psi_{P,U}, \operatorname{Ran} P) \to (\Psi_{P,U}, \operatorname{Ran} P)$ *is the identity morphism; and*

(ii) *if* $\tau : \chi_1 \to \chi_2$ *and* $\tau' : \chi_2 \to \chi_3$ *are two morphisms in* $\mathfrak{B}$, *then*

$$(\tau' \circ \tau)_* = \tau'_* \circ \tau_*.$$

Moreover, if χ_1 *and* χ_2 *are unitarily equivalent via a unitary similarity* τ, *then so are* Ψ_{P_1,U_1} *and* Ψ_{P_2,U_2} *via the induced unitary* τ_*.

A canonical model triple

It is natural to expect a converse of the ‘moreover’ part in the above result. However, unlike the forward direction, this model triple is not uniquely determined by the contractive analytic function. For example, one can check that both the unitaries

$$\left[\begin{array}{c|cc} 0 & 0 & 1 \\ \hline 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right], \left[\begin{array}{c|cc} 0 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right] : \begin{bmatrix} \mathbb{C} \\ \mathbb{C}^2 \end{bmatrix} \to \begin{bmatrix} \mathbb{C} \\ \mathbb{C}^2 \end{bmatrix}$$

serve as a unitary colligation for the contractive function $z \mapsto z^2$. Consequently, the function $z \mapsto z^2$ has two distinct model triples. There is, nevertheless, a canonical choice of a model triple for a contractive analytic function.

A canonical model triple

For an object $(\Psi, \mathcal{E})$ in $\mathfrak{C}$, consider the associated de Branges–Rovnyak reproducing kernel

$$K^{\Psi}(z, w) = \frac{I_{\mathcal{E}} - \Psi(z)\Psi(w)^*}{1 - z\bar{w}}. \tag{10}$$

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Let $\mathcal{H}$ be a Hilbert space and $g : \mathbb{D} \to \mathcal{B}(\mathcal{H}, \mathcal{E})$ be a function such that

$$K^{\Psi}(z, w) = g(z)g(w)^*. \tag{11}$$

This is the Kolmogorov decomposition of the kernel K^{Ψ}.

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Using the definition (10) of K^{Ψ} and after a rearrangement of the terms one arrives at

$$\langle e, f\rangle_{\mathcal{E}} + \langle \bar{w} g(w)^* e, \bar{z} g(z)^* f\rangle_{\mathcal{H}} = \langle \Psi(w)^* e, \Psi(z)^* f\rangle_{\mathcal{E}} + \langle g(w)^* e, g(z)^* f\rangle_{\mathcal{H}}$$

for every $z, w \in \mathbb{D}$ and $e, f \in \mathcal{E}$.

This readily implies that the map

$$\mathfrak{u} : \overline{\text{span}} \left\{ \begin{bmatrix} I_{\mathcal{E}} \\ \bar{z} g(z)^* \end{bmatrix} f : z \in \mathbb{D} \text{ and } f \in \mathcal{E} \right\} \to$$

$$\overline{\text{span}} \left\{ \begin{bmatrix} \Psi(z)^* \\ g(z)^* \end{bmatrix} f : z \in \mathbb{D} \text{ and } f \in \mathcal{E} \right\}$$

defined densely by

$$\mathfrak{u} : \sum_{j=1}^{N} \begin{bmatrix} I_{\mathcal{E}} \\ \bar{z}_j g(z_j)^* \end{bmatrix} f_j \mapsto \sum_{j=1}^{N} \begin{bmatrix} \Psi(z_j)^* \\ g(z_j)^* \end{bmatrix} f_j \qquad (12)$$

is a unitary.

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We wish to extend this partially defined unitary to whole of $\mathcal{E} \oplus \mathcal{H}$, which we can do if the orthocomplements of the domain and codomain of $\mathfrak{u}$ in $\mathcal{E} \oplus \mathcal{H}$ have the same dimension; if not, we can add an infinite dimensional Hilbert space say, $\mathcal{R}$ to $\mathcal{H}$ so that $\mathfrak{u}$ has a unitary extension to $\mathcal{E} \oplus \mathcal{H} \oplus \mathcal{R}$.

A canonical model triple

There is a *minimal* choice of the auxiliary Hilbert space $\mathcal{H}$, viz.,

$$\mathcal{H}_\Psi := \overline{\text{span}}\{g(z)^* e : z \in \mathbb{D} \text{ and } e \in \mathcal{E}\},$$

and that this is actually isomorphic to the defect space of M_Ψ^*.

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and that this is actually isomorphic to the defect space of M_Ψ^*.
Indeed, from the Kolmogorov decomposition (11) of K^Ψ, we see that

$$\langle (I_{H^2(\mathcal{E})} - M_\Psi M_\Psi^*)\mathbb{S}_w e, \mathbb{S}_z f\rangle_{H^2(\mathcal{E})} = \langle g(w)^* e, g(z)^* f\rangle_{\mathcal{H}_\Psi},$$

where $\mathbb{S}$ is the Szegö kernel for $\mathbb{D}$. This in particular implies that the map densely defined as

$$(I_{H^2(\mathcal{E})} - M_\Psi M_\Psi^*)^{\frac{1}{2}} \sum_{j=1}^{N} \mathbb{S}_{w_j} e_j \mapsto \sum_{j=1}^{N} g(w_j)^* e_j$$

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is a unitary.
For a contractive analytic function $(\Psi, \mathcal{E})$, denote by $\mathcal{F}_\dagger$ the minimal space containing $\mathcal{E} \oplus \mathcal{H}_\Psi$ to which the partially defined unitary $\mathfrak{u}$ as in (12) can be extended. Let $U_\dagger$ be a unitary operator on $\mathcal{F}_\dagger$ that extends $\mathfrak{u}$ and $P_\dagger$ be the orthogonal projection of $\mathcal{F}_\dagger$ onto $\mathcal{E}$.

A canonical model triple

Definition

A model triple $(\mathcal{F}_\dagger, P_\dagger, U_\dagger)$ obtained from a contractive analytic function $(\Psi, \mathcal{E})$ as above will be referred to as a *canonical model triple* for $(\Psi, \mathcal{E})$.

Theorem

If two contractive analytic functions $(\Psi_1, \mathcal{E}_1)$ *and* $(\Psi_2, \mathcal{E}_2)$ *are unitarily equivalent, then their canonical model triples are also unitarily equivalent.*

THANK YOU!