UM 102 - Mid Term

Maximum marks one can score in this paper is 40

1. If r_1 and r_2 denote the distances from a point (x,y) on an ellipse to its foci, show that the equation

$$r_1 + r_2 = \text{constant}$$

satisfied by these distances implies the relation

$$T \cdot \nabla (r_1 + r_2) = 0,$$

where T is the unit tangent to the curve. 5

Conclude from the above that the tangent makes equal angles with the lines joining (x,y) to the foci. 2

2. Determine a, b, c, d, e, f given that the vectors $(1,1,1)$, $(1,0,-1)$ and $(1,-1,0)$ are eigenvectors of the matrix 5

$$\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ d & e & f \end{pmatrix}$$

3. Let V be the linear space of all continuous functions on $(-\infty, \infty)$ such that the integral $\int_{-\infty}^{x} t f(t)\,dt$ exists for all x. Define T on V by

$$(Tf)(x) = \int_{-\infty}^{x} t f(t)\,dt.$$

Show that T is a linear transformation. 2

Prove that every negative λ is an eigenvalue of T and determine the eigenvectors corresponding to the eigenvalue λ. 5

4. Let V be the linear space of all real convergent sequences. Define $T: V \to V$ by

$$\{x_n\}_{n \geq 1} \to \{a - x_n\}_{n \geq 1}$$

where $a = \lim_{n \to \infty} x_n$. Prove that T is linear and describe the null space and range of T. 5

5. Let A be an $n \times n$ matrix containing an $r \times (n+1-r)$ zero submatrix for some $r \in \{1, \ldots, n\}$. Show that the determinant of A is zero. 5

6. A cylinder whose equation is $y = f(x)$ is tangent to the surface $z^2 + 2xz + y = 0$ at all points common to the two surfaces. Find $f(x)$. 5

7. Find values of the constants a, b, c such that the directional derivative of $f(x,y,z) = axy^2 + byz + cz^2x^3$ at the point $(1, 2, -1)$ has a maximum value of 64 in a direction parallel to the z-axis. 5

8. If $f(x,y) = [\sin(x^2+y^2)] / (x^2+y^2)$ for $(x,y) \neq (0,0)$, how must $f(0,0)$ be defined to make f continuous at the origin? 5

9. (a) Find a vector $V(x,y,z)$ normal to the surface

$$z = \sqrt{x^2+y^2} + (x^2+y^2)^{3/2}$$

at a point (x,y,z) of the surface, $(x,y,z) \neq (0,0,0)$. 5

(b) Find the cosine of the angle θ between $V(x,y,z)$ and the z-axis and determine the limit of $\cos\theta$ as $(x,y,z) \to (0,0,0)$. 5