UM 102 - Exercise Set 3

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[1] Consider the *Bernoulli* equation:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n \qquad , n \neq 1 \tag{1}$$

Use appropriate change of variable to convert the equation into a first order linear equation, and use it to solve the following:

(a)$y'' + ay = 2xy^2$

(b)$x^2y' - y^3 = xy$

[2]Consider the ODE $\frac{dx}{dt} + ax = r(t)$ where a is a positive constant and $\lim_{t\to\infty} r(t) = 0$.
Show that $\lim_{t\to\infty} x(t) = 0$.

[3]Let the Wronskian be defined for two functions u_1 and u_2 as follows:
$W(x) = u_1(x)u_2'(x) - u_2(x)u_1'(x)$ on an interval I.
Then:

(a)Let the Wronskian be 0 on an interval I. Then show that the ratio $\frac{u_2}{u_1}$ is a constant on the interval I. Hence, if $\frac{u_2}{u_1}$ is not constant on an interval I then we must have that W is non-zero for atleast one point in I.

(b)Calculate the derivative of the Wronskian.

(c)Now consider u_1 and u_2 are two solutions of the equation $y'' + ay + b = 0$, where a and b are constants. Then show that the Wronskian satisfies the equation $W' + aW = 0$.Thus, we have $W(x) = W(0)e^{-ax}$. Furthermore,let u_1 be not identically 0 on an interval. Then show that $W(0) = 0$ iff $\frac{u_2}{u_1}$ is a constant.

(d)Consider u_1 and u_2 as above and further assume that $\frac{u_2}{u_1}$ is non-constant. Then let $y = f(x)$ be another solution of the differentail equation $y'' + ay + b = 0$. Then show that there exist constants c_1 and c_2 such that:

$$c_1 u_1(0) + c_2 u_2(0) = f(0) \tag{2}$$

$$c_1 u_1'(0) + c_2 u_2'(0) = f'(0) \tag{3}$$

Now prove that every solution of the differential equation satisfies $f(x) = c_1 u_1 + c_2 u_2$ for appropriate constants c_1 and c_2