UM 102 - Exercise Set 5

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1. Consider a scalar field on $\mathbb{R}^n$ given by f($\mathbf{x}$)= $\mathbf{a} \cdot \mathbf{x}$, where $\mathbf{a}$ is a fixed vector.Compute $f'(\mathbf{x}; \mathbf{y})$ for arbitrary $\mathbf{x}$ and $\mathbf{y}$.

2. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Compute the derivative $f'(\mathbf{x}; \mathbf{y})$ for scalar function $f(\mathbf{x}) = \mathbf{x} \cdot T(\mathbf{x})$.

3. Compute the first-order partial derivatives of the following functions defined on $\mathbb{R}^2$:

   (a)$f(x, y) = x^2 + y^2 sin(x.y)$

   (b)$f(x, y) = \sqrt{x^2 + y^2}$

   (c)$f(x, y) = \frac{x}{\sqrt{x^2+y^2}}$

   (d)$f(x, y) = \frac{x+y}{x-y}$ $\quad ,x \neq y$

4. Check for the functions described, whether the mixed partials are equal, i.e, whether $D_1(D_2 f) = D_2(D_1 f)$:

   (a) All the functions described in [3] above.

   (b) $f(x, y) = x^4 + y^4 - 4x^2y^2$

   (c)$f(x, y) = log(x^2 + y^2)$

   (d)$f(x, y) = \frac{cos(x^2)}{y}$

5. (a) Consider an $n-$ball $B(\mathbf{a})$. Let $f'(\mathbf{x}; \mathbf{y}) = 0$ for every $\mathbf{x}$ in $B(\mathbf{a})$, and every $\mathbf{y}$.Show that f is constant on $B(\mathbf{a})$.
   [HINT: Use the mean value theorem].

   (b) Does the conclusion still hold if the condition above is only true for a fixed $\mathbf{y}$?

6. A set S in $\mathbb{R}^n$ is said to be convex if for $\mathbf{a}, \mathbf{b} \in S$, and for $0 \leq t \leq 1$, we have $t\mathbf{a} + (1-t)\mathbf{b} \in S$,i.e, the line joining $\mathbf{a}$ and $\mathbf{b}$ lies in S.

   (a)Show that every $n-$ball $B(\mathbf{a})$ is convex.

   (b) Let $f'(\mathbf{x}; \mathbf{y}) = 0$ for every $\mathbf{x}$ in S, and every $\mathbf{y}$, where S is an open convex set. Show that f is constant on S.

7. Find the gradient of the following functions:

   (a) $f(x, y) = x^2 + y^2 sin(x.y)$

   (b) $f(x, y) = e^x cos(y)$

   (c) $f(x, y, z) = x^2 - y^2 + 2z^2$

   (d) $f(x, y, z) = log(x^2 + 2y^2 - 3z^2)$

8. Consider the function $f(x, y) = 3x^2 + y^2$ defined on the set $S = \{(x, y) : x^2 + y^2 = 1\}$. Find the points in S and the directions for which the directional derivative is maximum.

9. Find values of the constants a,b, and c such that the directional derivative of $f(x, y, z) = a.x.y^2 + b.y.z + c.z^2.x^3$ at the point (1, 2, —1) has a maximum value of 64 in a direction parallel to the z-axis.

10. Let f and g denote scalar fields that are differentiable on an open set S. Derive the following properties of the gradient:

    (a) grad f = 0 if f is constant on S.

    (b) grad (f + g) = grad(f) + grad(g).

    (c) grad (cf) = c grad (f), if c is a constant.

    (d) grad (fg) =f grad(g) +g.grad(f).

11. In $\mathbb{R}^3$ let $\mathbf{r}$(x,y,z) = x$\mathbf{i}$ + y$\mathbf{j}$ + z$\mathbf{k}$, and let r(x,y,z) = $||\mathbf{r}(x, y, z)||$.

    (a) Show that $\nabla r(x, y, z)$ is a unit vector in the direction of $\mathbf{r}$(x, y, z).

    (b) Show that $\nabla(r^n) = nr^{n\ 2}\mathbf{r}$, if n is a positive integer.

    (c) Find a scalar field f such that ∇f = r.

12. Let $B(\mathbf{a})$ be $n-$ball. Show that if x is point of minimum in $B(\mathbf{a})$, then $\nabla f(x) = 0$.

13. Do the conclusions of [6] still hold if the condition: "$f'(\mathbf{x}; \mathbf{y}) = 0$" is replaced with the condition:" grad(f)(x)=0 , for every x"?

14. Consider the following transformation $(x, y) \to (r, \theta)$, where $x = rcos(\theta)$ and $y = rsin(\theta)$.

    (a) Find the Jacobean of this tranformation. Use this to find D_r, and D_θ in terms of D_x and D_y.

    (b) Using the above, what do the following equations give in polar coordinates:

$$D_x u = D_y v \quad (1)$$

$$D_y u = -D_x v \quad (2)$$