UM-102 Calculus and Linear Algebra
Problem Set-2
Tirthankar Bhattacharyya

(1) Apply the Gauss-Jordan elimination process to both of the following systems. If a solution exists, determine the general solution.
(i) $3x + 2y + z = 1, 5x + 3y + 3z = 2, x + y - z = 1$ and
(ii) $3x + 2y + z = 1, 5x + 3y + 3z = 2, 7x + 4y + 5z = 3, x + y - z = 0$.

(2) Prove that the system $x + y + 2z = 2, 2x - y + 3z = 2, 5x - y + az = 6$, has a unique solution if $a \neq 8$. Find all solutions when $a = 8$.

(3) Compute the determinant of each of the following matrices by transforming each of them to an upper triangular matrix.

(i) $\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$, (ii) $\begin{bmatrix} 1 & t & t^2 & t^3 \\ t & 1 & t & t^2 \\ t^2 & t & 1 & t \\ t^3 & t^2 & t & 1 \end{bmatrix}$, (iii) $\begin{bmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^3 & b^3 & c^3 & d^3 \end{bmatrix}$.

(4) Let $A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$, where a_1, a_2, a_3 are rows of the matrix A. Find $det \begin{bmatrix} a_1 + a_2 \\ a_2 + a_3 \\ a_3 + a_1 \end{bmatrix}$.

(5) For each of the following statements about square matrices, give a proof or exhibit a counter example.
(i) $det(A + B) = detA + detB$
(ii) $det\{(A + B)^2\} = \{det(A + B)\}^2$
(iii) $det\{(A + B)^2\} = det(A^2 + 2AB + B^2)$
(iv) $det\{(A + B)^2\} = det(A^2 + B^2)$.

(6) Let $A = [a_{ij}]$ be a lower triangular matrix (all entries above the diagonal are zero, i.e. $a_{ij} = 0$ for all $i < j$). Using the axiom of a determinant function and properties derived, show that $detA = a_{11}a_{22} \cdots a_{nn}$. Note: Do not use the standard method of finding the determinant using the determinants of lower order.

(7) (a)Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ a & b & c & d \\ e & f & g & h \end{bmatrix}$. Prove that $detA = det \begin{bmatrix} c & d \\ g & h \end{bmatrix}$.

(b) State and prove a generalization of above problem for $n \times n$ matrices.

(8) (a) Let $A = \begin{bmatrix} x & y & 0 & 0 \\ z & w & 0 & 0 \\ a & b & c & d \\ e & f & g & h \end{bmatrix}$. Prove that $detA = det \begin{bmatrix} x & y \\ z & w \end{bmatrix} det \begin{bmatrix} c & d \\ g & h \end{bmatrix}$.

(b) State and prove a generalization of above problem for $n \times n$ matrices of the form

$$A = \begin{bmatrix} B & 0 \\ C & D \end{bmatrix},$$

where B, C, D denote square matrices and 0 denotes a matrix of zeros.

(9) Let $a_{ij}(x)$ be differentiable in (a, b). Define the matrix $A(x) = [a_{ij}(x)]$ for each $x \in (a, b)$. Let $A_i(x)$ be the matrix obtained from $A(x)$ by replacing the i-th row by the derivative of the i-th row. Then show that

$$\frac{d}{dx}(det(A(x))) = \sum_{i=1}^{n} det(A_i(x)).$$

(10) Deduce the following: Consider the Wronskian matrix

$$W(x) == \begin{bmatrix} w_1(x) & w_2(x) & \cdots & w_n(x) \\ w_1'(x) & w_2'(x) & \cdots & w_n'(x) \\ \vdots & \vdots & & \vdots \\ w_1^{(n-1)}(x) & w_2^{(n-1)}(x) & \cdots & w_n^{(n-1)}(x) \end{bmatrix} = \begin{bmatrix} w(x) \\ w'(x) \\ \vdots \\ w^{(n-1)}(x) \end{bmatrix},$$

where $w(x) = (w_1(x), w_2(x), \cdots, w_n(x))$ etc. Show that

$$\frac{d}{dx}(det(W(x))) = \begin{bmatrix} w(x) \\ w'(x) \\ \vdots \\ w^{(n-2)}(x) \\ w^n(x) \end{bmatrix}.$$

(11) Determine whether the following sets of vectors are linearly dependent or independent.
(i) $A_1 = (1, -1, 0), A_2 = (0, 1, -1), A_3 = (2, 3, -1)$,
(ii)$A_1 = (1, 0, 0, 0, 1), A_2 = (1, 1, 0, 0, 0), A_3 = (1, 0, 1, 0, 1), A_4 = (1, 1, 0, 1, 1), A_5 = (0, 1, 0, 1, 0)$.