

1. Suppose that V is just a vector space and has no inner product as of yet, but W is an inner product space with inner product $\langle \bullet, \bullet \rangle_W$. Let $T: V \rightarrow W$ be a linear transformation. Does $\langle x, y \rangle_V := \langle T(x), T(y) \rangle_W$ define an inner product on V ? (Hint: No.) What additional property can we place on T to ensure that this does define an inner product on V ?
2. Let $L > 0$. Consider the inner product space V consisting of all continuous functions from $[-L, L]$ to \mathbb{R} with the inner product defined by $\langle f, g \rangle = \int_{-L}^L fg$. Let

$$\beta = \left\{ \cos\left(\frac{n\pi x}{L}\right) \mid n \in \mathbb{N} \right\} \cup \left\{ \sin\left(\frac{n\pi x}{L}\right) \mid n \geq 1 \right\}$$

Of course β is a subset of V . Show that β is orthogonal (i.e. for each pair $x \neq y$ of distinct vectors in β we have $\langle x, y \rangle = 0$). Find a formula for the length of each vector in β .

3. Let V and V' be inner product spaces. Let $T: V \rightarrow V'$ be an isomorphism of vector spaces that is also inner product preserving. Explain why T^{-1} is also inner product preserving. Let W be a subset of V . Show that $W^\perp = T^{-1}(T(W)^\perp)$
4. Two inner product spaces are said to be isomorphic (as inner product spaces) if there is a linear transformation between them that is 1-1, onto, and inner product preserving. Prove that if V is an inner product space of dimension n , then V and \mathbb{R}^n are isomorphic as inner product spaces too.
5. Let β be a (finite) orthonormal basis for the inner product space V and let γ be a (finite) orthonormal basis for the inner product space W . Prove that a linear transformation $T: V \rightarrow W$ is inner product preserving iff $[T]_{\beta}^{\gamma}$ is a matrix with orthonormal columns.
6. Let V be the inner product space of all continuous functions from $[-1, 1]$ to \mathbb{R} with inner product $\langle f, g \rangle = \int_{-1}^1 fg$. Let W_e denote the subspace of V consisting of all even functions and similarly let W_o denote all the odd functions. Prove that $(W_e)^\perp = W_o$. (Suggestion: Show that every function in V can be written as the sum of an even and an odd function.)