1. Suppose that V is just a vector space and has no inner product as of yet, but W is an inner product space with inner product $\langle \bullet, \bullet \rangle_W$. Let $T\colon V \to W$ be a linear transformation. Does $\langle x, y \rangle_V := \langle T(x), T(y) \rangle_W$ define an inner product on V? (Hint: No.) What additional property can we place on T to ensure that this does define an inner product on V?

2. Let $L > 0$. Consider the inner product space V consisting of all continuous functions from $[-L, L]$ to $\mathbb{R}$ with the inner product defined by $\langle f, g \rangle = \int_{-L}^{L} fg$. Let

$$\beta = \{\cos(\frac{n\pi x}{L}) \mid n \in \mathbb{N}\} \cup \{\sin(\frac{n\pi x}{L}) \mid n \geq 1\}$$

Of course β is a subset of V. Show that β is orthogonal (i.e. for each pair $x \neq y$ of distinct vectors in β we have $\langle x, y \rangle = 0$). Find a formula for the length of each vector in β.

3. Let V and V' be inner product spaces. Let $T\colon V \to V'$ be an isomorphism of vector spaces that is also inner product preserving. Explain why T^{-1} is also inner product preserving. Let W be a subset of V. Show that $W^{\perp} = T^{-1}(T(W)^{\perp})$

4. Two inner product spaces are said to be isomorphic (as inner product spaces) if there is a linear transformation between them that is 1-1, onto, and inner product perserving. Prove that if V is an inner product space of dimension n, then V and $\mathbb{R}^n$ are isomorphic as inner product spaces too.

5. Let β be a (finite) orthonormal basis for the inner product space V and let γ be a (finite) orthonormal basis for the inner product space W. Prove that a linear transformation $T\colon V \to W$ is inner product preserving iff $[T]_\beta^\gamma$ is a matrix with orthonormal columns.

6. Let V be the inner product space of all continuous functions from $[-1, 1]$ to $\mathbb{R}$ with inner product $\langle f, g \rangle = \int_{-1}^{1} fg$. Let W_e denote the subspace of V consisting of all even functions and similarly let W_o denote all the odd functions. Prove that $(W_e)^{\perp} = W_o$. (Suggestion: Show that every function in V can be written as the sum of an even and an odd function.)