

1. Suppose that  $V$  is just a vector space and has no inner product as of yet, but  $W$  is an inner product space with inner product  $\langle \bullet, \bullet \rangle_W$ . Let  $T: V \rightarrow W$  be a linear transformation. Does  $\langle x, y \rangle_V := \langle T(x), T(y) \rangle_W$  define an inner product on  $V$ ? (Hint: No.) What additional property can we place on  $T$  to ensure that this does define an inner product on  $V$ ?
2. Let  $L > 0$ . Consider the inner product space  $V$  consisting of all continuous functions from  $[-L, L]$  to  $\mathbb{R}$  with the inner product defined by  $\langle f, g \rangle = \int_{-L}^L fg$ . Let

$$\beta = \left\{ \cos\left(\frac{n\pi x}{L}\right) \mid n \in \mathbb{N} \right\} \cup \left\{ \sin\left(\frac{n\pi x}{L}\right) \mid n \geq 1 \right\}$$

Of course  $\beta$  is a subset of  $V$ . Show that  $\beta$  is orthogonal (i.e. for each pair  $x \neq y$  of distinct vectors in  $\beta$  we have  $\langle x, y \rangle = 0$ ). Find a formula for the length of each vector in  $\beta$ .

3. Let  $V$  and  $V'$  be inner product spaces. Let  $T: V \rightarrow V'$  be an isomorphism of vector spaces that is also inner product preserving. Explain why  $T^{-1}$  is also inner product preserving. Let  $W$  be a subset of  $V$ . Show that  $W^\perp = T^{-1}(T(W)^\perp)$ .
4. Two inner product spaces are said to be isomorphic (as inner product spaces) if there is a linear transformation between them that is 1-1, onto, and inner product preserving. Prove that if  $V$  is an inner product space of dimension  $n$ , then  $V$  and  $\mathbb{R}^n$  are isomorphic as inner product spaces too.
5. Let  $\beta$  be a (finite) orthonormal basis for the inner product space  $V$  and let  $\gamma$  be a (finite) orthonormal basis for the inner product space  $W$ . Prove that a linear transformation  $T: V \rightarrow W$  is inner product preserving iff  $[T]_\beta^\gamma$  is a matrix with orthonormal columns.
6. Let  $V$  be the inner product space of all continuous functions from  $[-1, 1]$  to  $\mathbb{R}$  with inner product  $\langle f, g \rangle = \int_{-1}^1 fg$ . Let  $W_e$  denote the subspace of  $V$  consisting of all even functions and similarly let  $W_o$  denote all the odd functions. Prove that  $(W_e)^\perp = W_o$ . (Suggestion: Show that every function in  $V$  can be written as the sum of an even and an odd function.)