LECTURE - 1 (HOMOGENEOUS POLYNOMIALS AND PROJECTIVE MANIFOLDS)

Recall that one produce submanifolds of $\mathbb{CP}^n$ using homogeneous polynomials. Indeed, here are examples of such submanifolds :

(1) $\sum a_i X_i = 0$. This is called a hyperplane. That this is a submanifold is easy to see : If $X_j \neq 0$, then we may divide by X_j, consider coordinates $z_{i,j} = \frac{X_i}{X_j}$ and see that since this is a linear relation, we can solve for one of these in terms of the others in a holomorphic manner.

(2) More generally, if $F(X_0, \ldots, X_n) = 0$ is a homogeneous polynomial such that $\nabla F \neq 0$ on the zero locus, then this defines a submanifold of $\mathbb{CP}^n$. Indeed, suppose we choose a coordinate chart where $X_0 \neq 0$ and assume that $\frac{\partial F}{\partial X_j} \neq 0$. Defining $z_i = \frac{X_i}{X_0}$ we see that $f(z_1, \ldots, z_n) = F(1, z_1, \ldots, z_n) = 0$. Taking derivatives we get $\frac{\partial f}{\partial z_j} = \frac{\partial F}{\partial X_j} \neq 0$. Therefore, by the implicit function theorem, we are done. (Suppose $j = 0$, and that $\frac{\partial f}{\partial z_i} = 0$ for all other i. This situation is not possible. (Why ?))

(3) Likewise, $\sum X_i^2 = 0$, $\sum X_i = 0$ defines a complex codimension-2 submanifold. (This can be easily generalised to a bunch of homogeneous polynomials with independent derivatives.)

Now what are polynomials like X_0, $X_1^2 + X_2^2$ etc maps to ? They are surely not holomorphic functions on $\mathbb{CP}^n$. Let's write them down in local coordinates. Take the degree one polynomial X_1. Now suppose we choose a coordinate chart $U_0 : X_0 \neq 0$. Then $z_i = \frac{X_i}{X_0}$ are local coordinates on U_0, i.e., U_0 is homeomorphic to $\mathbb{C}^n$. Now the polynomial $X_1 = z_1 X_0$, i.e., it is "function" $z_1 : U_0 \to \mathbb{C}$. If we choose another coordinate chart like $U_j : X_j \neq 0$, then $X_1 = w_1 X_j$. Note that w_1 and z_1 are not the same on $U_j \cap U_0$ but are related by multiplication with $\frac{w_1}{z_1}$. Thus morally speaking, X_1 should be thought of as a map, not to $\mathbb{C}$ but to a manifold (which we shall denote as $\mathcal{O}(1)$) defined as $\frac{\cup_i U_i \times \mathbb{C}}{On\ U_i \cap U_j\ ,(p,v_i) \equiv (p, g_{ij} v_j)}$ where $g_{ij} = \frac{X_j}{X_i}$. These g_{ij} are obviously holomorphic functions from $U_i \cap U_j$ to $\mathbb{C}^*$. They also satisfy $g_{ij} = g_{ji}^{-1}$ and $g_{ij} g_{jk} g_{kl} = 1$. This manifold $\mathcal{O}(1)$ is known to algebraic geometers as the "Hyperplane line bundle on $\mathbb{CP}^n$". The g_{ij} are called the "transition functions" of the line bundle.

The manifold $\mathcal{O}(1)$ is a curious object. It admits an obvious "projection" map π to $\mathbb{CP}^n$ such that $\pi^{-1}(p)$ is $\mathbb{C}$, i.e., a 1-D complex vector space. So, in a sense, it consists of complex lines, varying holomorphically, parametrised by $\mathbb{CP}^n$. This is an example of a holomorphic line bundle. In general, a holomorphic line bundle L on a complex manifold X is simply a complex manifold $L = \frac{\cup_\alpha U_\alpha \times \mathbb{C}}{(p,v_\alpha) \equiv (p, g_{\alpha\beta} v_\beta)}$ where U_α is a collection of open sets on X such that $X = \cup_\alpha U_\alpha$ (they are called "trivialising open sets of L"), $g\alpha\beta : U_\alpha \cap U_\beta \to \mathbb{C}^*$ is a collection of holomorphic functions (called "the transition functions of L") satisfying $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ and $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$.

Exercise 0.1. *Prove that*

(1) *The holomorphic line bundle L is actually a complex manifold of dimension $n + 1$.*

(2) *Also prove that there is a holomorphic projection map $\pi : L \to X$ such that $\pi^{-1}(p) = \mathbb{C}$, i.e., a 1-D vector space. These vector spaces are called the "fibres" of the line bundle.*

(3) *Moreover, prove that around every point $p \in X$, there is an open set U such that $\pi^{-1}(U)$ is biholomorphic to $U \times \mathbb{C}$ with the map preserving fibres and the biholomorphism being linear on the fibres. (This is called being "locally trivial".)*

(4) *(Optional) Prove that every complex manifold L that satisfies the second and third points above is actually biholomorphic to the holomorphic line bundle we defined (with the biholomorphism preserving fibres and being linear on them).*

A holomorphic function $s : X \to L$ such that $\pi \circ s(p) = p$ is called a holomorphic section of L. For instance, $X_0, X_1, \ldots, X_n$ are holomorphic sections of $\mathcal{O}(1)$ over $\mathbb{CP}^n$. We say that $s : U \to \mathbb{C}$ provides a local trivialisation for L over U if $s \neq 0$ anywhere on U, i.e., using s one can show that L restricted to U is actually isomorphic to the trivial line bundle $U \times \mathbb{C}$.

Other than $\mathcal{O}(1)$ on $\mathbb{CP}^n$, what examples of holomorphic line bundles can we come up with ? A stupid example is $X \times \mathbb{C}$. This is (rightly) called the trivial line bundle over X. Here are some constructions of new line bundles from two given ones V and W on X with transition functions g and h.

(1) V^* is the dual line bundle whose fibres are the duals of V_p and whose transition functions are $g^{-1}_{\alpha\beta}$.
(2) $V \otimes W$ is a line bundle whose fibres are tensor products of the vector spaces and whose transition functions are the product of the matrices g and h.
(3) Suppose $f : N \to M$ is a holomorphic map then $f^*(V)$ (called the "pullback of V") is a vector bundle over N with the same fibres but with transition functions $f^*g_{\alpha\beta} = g_{\alpha\beta} \circ f$. For example, if $i : N \subset M$ is a complex submanifold of M, then $i^*(V)$ is called the restriction of V to N. The transition functions are simply restrictions.

The above definition of $\mathcal{O}(1)$ seems too contrived. Here is a more pleasant geometric definition of the Tautological line bundle $\mathcal{O}(-1)$.

Definition 0.2. The total space of the tautological line bundle is a subset of $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ consisting of $([X_0 : X_1 : \ldots], v_0, v_1, \ldots, v_{n+1})$ such that $\vec{v} = \mu\vec{X}$ for some complex number μ. The projection map is $\pi([X_0 : X_1 \ldots], \vec{v}) = [X_0 : X_1 \ldots]$. In other words, on the space of lines through the origin, at every line, simply choose the 1-D vector space represented by that line. The dual bundle $\mathcal{O}(1)$ consists of linear functionals on each of those lines.

Exercise 0.3. *Prove that the tautological line bundle as defined above is indeed the dual of $\mathcal{O}(1)$ as defined earlier.*

Why is it denoted as $\mathcal{O}(1)$? The reason is that homogeneous polynomials of degree 1 are holomorphic sections of this bundle. (Indeed, we constructed this bundle so that precisely this happens.)

In fact, something stronger is true : All holomorphic sections of $\mathcal{O}(1)$ correspond to homogeneous degree-1 polynomials. (This is an example of the slogan of Serre's GAGA : "Analytic and algebraic geometry coincide on the projective space.")

Its proof is as follows :

Homogeneous degree-1 polynomials correspond to holomorphic sections of $\mathcal{O}(1)$: Indeed, given $F(X_0, X_1, \ldots) = \sum a_i X_i$ where at least one $a_j \neq 0$, we have already seen that these correspond to sections of $\mathcal{O}(1)$ (indeed $\mathcal{O}(1)$ was defined in a such a way that X_1 corresponds to a section. You can easily verify that $\sum a_i X_i$ also canonically defines a section). However, we shall do this in another way, i.e., by interpreting $\mathcal{O}(1)$ as the dual of the tautological line bundle $\mathcal{O}(-1)$. A section of $\mathcal{O}(1)$ is supposed to be a linear functional at every point $[X_0 : X_1 : X_2 \ldots]$ on the corresponding 1-D vector space consisting of vectors $\vec{v}$ lying along the line defined by $[X_0 : X_1 \ldots]$. In other words, define $\langle s_F([X_0 : X_1 \ldots]), \vec{v}\rangle = \sum a_i v_i$. This is holomorphic. Indeed, on $U_0 : X_0 \neq 0$ for instance (the other U_j behave similarly), $[X_0 : X_1 \ldots] = [1 : z_1 : z_2 \ldots]$ and $\vec{v} = v_0(1, z_1, z_2 \ldots)$, $\langle s_F(z_1, z_2, \ldots), v_0(1, z_1, \ldots) = v_0(a_1 + a_2 z_2 + \ldots)$. Thus, locally, $\langle s_F, \vec{v}\rangle$ behaves linearly in $\vec{v}$ and holomorphically in z as per definition.

All holomorphic sections correspond to homogeneous polynomials : Suppose s is a section of $O(1)$. Then at every point $[X_0 : X_1 : \ldots]$, $s([X_0 : X_1 \ldots])$ is a linear functional that takes $\vec{v} = \mu\vec{X}$ and spits out a complex number. This means that we can talk of a holomorphic function $F([X_0 : \ldots, X_n], v_0, v_1 \ldots) = \langle s([X_0 : X_1 \ldots]), v\rangle$ such that $F([X_0 : \ldots], \lambda v_0, \lambda v_1 \ldots) = \lambda F([X_0 : X_1 \ldots], \vec{v})$. Moreover, since $\vec{v} = \mu\vec{X}$, the previous function is actually simply a holomorphic function $F(X_0, X_1, \ldots, X_n)$ on $\mathbb{C}^{n+1} - \vec{0}$ such that $F(\lambda X_0, \lambda X_1, \ldots) = \lambda F(X_0, \ldots, X_n)$. By Hartog's theorem this extends to all of $\mathbb{C}^{n+1}$. Moreover, $\frac{\partial F}{\partial X_i}$ is a homogeneous function of degree 0. Thus it is a constant equal to its value at the origin. Thus F is linear. $\square$

Exercise 0.4. *Define $O(k)$ as the tensor product of $O(1)$ with itself k-times. Prove that its holomorphic sections correspond to degree k homogeneous polynomials.*

The bottom line is that there are holomorphic line bundles on $\mathbb{CP}^n$ (and thus on its submanifolds) that have lots of holomorphic sections and that the "homogeneous coordinates" $X_0, X_1 \ldots$ on $\mathbb{CP}^n$ are secretly sections of a certain line bundle, namely, $O(1)$.

A natural question is "Which compact complex manifolds arise as submanifolds of $\mathbb{CP}^n$? (such things are called "projective varieties")" The answer to this question is provided by the Kodaira embedding theorem. As an application of the Kodaira embedding theorem, it turns out that if you choose a complex torus at random, then almost surely it will NOT be projective. On the other hand, all Riemann surfaces (1-D complex manifolds) are projective.