LECTURE 7 - THE CALABI CONJECTURE(S)

At this point we note that the Ricci curvature is quite useful in Riemannian geometry. In particular, we have the Bonnet-Myers theorem : If (M, g) is a complete Riemannian manifold satisfying $Ric \geq Kg$ where $K > 0$ is a constant, then M is compact. As a consequence, the universal cover of M is also compact and hence the fundamental group is finite. This theorem motivates the following conjecture due to Calabi in the Kähler case (let's call this "Calabi's Ricci conjecture"):

Let (M, ω) be a compact Kähler manifold and $\eta \in [c_1(M, K_M^*)]$. Then there exists a unique Kähler metric $\omega' \in [\omega]$ such that $Ricc(g') = 2\pi c_1(g') = \eta$.

Assuming this conjecture to be true (was shown by Yau), if K_M^* is a positive line bundle (which for instance is the case when $d < n+1$ for a degree-d hypersurface of $\mathbb{CP}^n$), then there exists a Kähler metric of positive Ricci curvature and hence the fundamental group is finite !
Yau also showed that anything diffeomorphic to $\mathbb{CP}^2$ is biholomorphic to it using this version of Calabi's conjecture.

A related natural question is the existence of Riemannian metrics g satisfying $Ric(g) = \lambda g$ where λ is a constant. Such metrics are called Einstein metrics (because in the Lorentzian context, such metrics represent gravity in a region of space where there is no ordinary matter/energy but can have dark energy). These metrics are difficult to find and if found, they are typically not unique. Their study is quite popular in Riemannian geometry. In the Kähler case Einstein metrics are not only much better behaved but also, their existence leads to algebro-geometric consequences (for the interested, search for "Bogomolov-Miyaoka-Yau inequality" on google). There is a Calabi Conjecture for the existence of Kähler-Einstein metrics too. However, there is a small subtlety here. Note that if $2\pi c_1(g) = \lambda g$, then $[c_1]$ has a sign, i.e., either $[c_1(K_M^*)] = [0]$ or it has a positive or a negative representative. (Note that this is not a trivial condition. It is not at all clear that the De Rham cohomology class of a $(1,1)$-form has a representative that locally gives a positive-definite or a negative-definite matrix.)

Calabi's KE conjecture : Let M be a compact Kähler manifold whose $c_1(M) = [c_1(K_M^*)]$ is either $[0]$ or has a positive or a negative representative. Then

(1) If $[c_1(M)] = [0]$, in every Kähler class $[\omega]$ there exists a unique Kähler metric $\omega' \in [\omega]$ such that $Ric(\omega') = 0$.
(2) If $[c_1(M)] < 0$ or $[c_1(M) > 0$, then there exists a unique Kähler metric ω such that $Ric(\omega) = \omega$ (if $c_1 > 0$) or $Ric(\omega) = -\omega$ if $c_1 < 0$.

This conjecture was studied by Aubin and Yau originally. They proved it in the $c_1 = 0, < 0$ cases. It is unfortunately FALSE in the $c_1 > 0$. A necessary and sufficient condition was discovered (by a bunch of people including but not limited to Chen, Donaldson, Sun, and Tian) and the mystery was resolved only in 2012. After that a few more proofs were given (simplifying the original ones) - in particular - our very own Ved (along with Gabor Szekelyhidi) gave one.

1. The Poisson ODE and Fourier analysis

Suppose we want to solve the ODE $u'' = f$ for a 2π periodic smooth function u where f is a 2π-periodic smooth function, then

$$u'(x) = u'(0) + \int_0^x f(t)dt \tag{1.1}$$

$u(x) = u(x+2\pi)$ implies that $u'(x) = u'(x+2\pi)$ (in fact they are equivalent if $u(0) = u(2\pi)$). Thus $\int_0^{2\pi} f(t)dt = 0$. This is a necessary and sufficient (by the periodicity of f, $\int_x^{x+2\pi} f(t)dt = \int_0^{2\pi} f(t)dt$) condition. (Smoothness is guaranteed by the fundamental theorem of calculus.)

In other words, there is a unique-upto-a-constant smooth periodic solution of the ODE if and only if f is smooth, periodic, and satisfies $\int_0^{2\pi} f(t)dt = 0$. Interestingly enough, denoting the vector space of smooth 2π-periodic functions as C^∞, the map $T : C^\infty \to C^\infty$ given by $T(u) = u''$ has kernel precisely the constants. Moreover, equipping this vector space with the inner product $\langle u, v\rangle = \int_0^{2\pi} uvdx$, we see that $T = T^*$ and $T(u) = f$ if and only if f is orthogonal to $ker(T^*) = ker(T)$. This is very similar to finite-dimensional linear algebra. Moreover, by the fundamental theorem of calculus, if f is k-times continuously differentiable (will be denoted as C^k from now on), then u is C^{k+2}.

The above mentioned observations are not coincidences. Later on, we will see that many PDE (the so-called elliptic PDE) satisfy similar properties. However, to prove such things, we cannot rely on a direct formula for the solution unlike the case of ODE. So we need a more abstract, theoretical method.

Thinking naively (like an engineer or a physicist) we write the Fourier series $u = \sum_{k=-\infty}^{\infty} \hat{u}(k)e^{ikx}$ where $\hat{u}(k) = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-ikx}dx$ and likewise for f. Then we see that

$$\hat{u}(k)k^2 = -\hat{f}(k) \tag{1.2}$$

In other words, there is a (formal) solution if and only if $\hat{f}(0) = 0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)dx$. In this case, $\hat{u}(0)$ is a free parameter and hence the solution is unique upto a constant. Moreover, since as sharp changes in music (think of opera music) correspond to very shrill sounds, if the high-frequency Fourier components are "small", then the function is very "smooth" (melodious notes are not too shrill). Since $\hat{u}(k) = \frac{\hat{f}(k)}{k^2}$, u behaves more smoothly than f does. So if f is a smooth function, we expect u to be so as well.

To make things rigorous, firstly, notice that the Fourier coefficients make sense for any integrable function. The convergence of Fourier series is a subtle phenomenon though. For example, there exist continuous functions whose Fourier series do not convergence pointwise at some points. [1] Nonetheless, we have the following useful results.

1. Riesz-Fischer : A measurable function on $[0, 2\pi]$ is in L^2 if and only if its Fourier series converges in the L^2 norm to it. Moreover, if a_k is in l^2, then $\sum a_k e^{ikx}$ converges in L^2.
2. Parseval-Plancherel : The Fourier series transform is an isometric isomorphism between $L^2([0, 2\pi])$ and l^2.

[1] Already this is beginning to hint that expecting results like "If f is C^k, then u is C^{k+2}" is a bad idea from the Fourier-analytic point of view. In fact for PDE, this expectation is false.

(3) Let $C^{0,\alpha}$ $(0 < \alpha < 1)$ consist of all Hölder continuous 2π-periodic functions g, i.e., periodic functions g such that $|g(x) - g(y)| \leq C|x - y|^{\alpha}$ for all x, y. Note that if f is in C^1, then f is Hölder continuous.
Theorem : If $f \in C^{0,\alpha}$ then $|\hat{f}(k)| \leq \frac{K}{|k|^{\alpha}}$ $\forall$ $|k| \geq 1$.

Proof.

$$2\pi \frac{\widehat{f(x+h)}(k) - \hat{f}(k)}{h^{\alpha}} = \int_0^{2\pi} \frac{f(x+h) - f(x)}{h^{\alpha}} e^{-ikx} dx$$

$$\Rightarrow \left|\frac{\widehat{f(x+h)}(k) - \hat{f}(k)}{h^{\alpha}}\right| \leq C \tag{1.3}$$

Now

$$\left|\int_0^{2\pi} \frac{f(x+h) - f(x)}{h^{\alpha}} e^{-ikx} dx\right| = \left|\int_h^{2\pi+h} \frac{f(y)}{h^{\alpha}} e^{-ik(y-h)} - \int_0^{2\pi} \frac{f(x)}{h^{\alpha}} e^{-ikx} dx\right|$$

$$= \left| - \int_0^{h} \frac{f(y)}{h^{\alpha}} e^{-ik(y-h)} dy + \int_{2\pi}^{2\pi+h} \frac{f(y)}{h^{\alpha}} e^{-ik(y-h)} dy + \frac{1}{h^{\alpha}} \int_0^{2\pi} e^{-ikx} f(x)(e^{ikh} - 1) dx\right|$$

$$= \left|\frac{1}{h^{\alpha}} \int_0^{2\pi} e^{-ikx} f(x)(e^{ikh} - 1) dx\right| = |\hat{f}(k)| \frac{|e^{ikh} - 1|}{h^{\alpha}} \tag{1.4}$$

Take $h = \frac{1}{k}$. Using 1.3 and 1.4 we see that $|\hat{f}(k)| \leq \frac{K}{k^{\alpha}}$.
As for uniform convergence, □

(4) Theorem : If $f \in C^{0,\alpha}$, the Fourier series converges uniformly to f.
(5) Theorem : If $f \in C^1$ then $\hat{f'} = ik\hat{f}(k)$. This holds for higher derivatives too.

Proof.

$$\hat{f'} = \frac{1}{2\pi} \int_0^{2\pi} f'(x) e^{-ikx} dx = -\frac{1}{2\pi} \int_0^{2\pi} f(x) (e^{-ikx})' dx = \frac{1}{2\pi} \int_0^{2\pi} ikf(x) e^{-ikx} dx = ik\hat{f}(k) \tag{1.5}$$

□

(6) Theorem : If f is smooth, then the Fourier coefficients are rapidly decaying (decay faster than any polynomial). Also the Fourier series of f and its derivatives converge uniformly. Conversely, if a_k are rapidly decaying, then they are the Fourier coefficients of a smooth function (with convergence being uniform).

Proof. If f is smooth, then $\hat{f^{(l)}}(k) = (ik)^l \hat{f}(k)$. Since $\hat{f^{(l)}}(k)$ is bounded, $\hat{f}(k)$ is rapidly decaying. By one of the earlier theorems, the convergence is uniform.

If $|a_k| \leq C_l |k|^{-l}$, then by the Weierstrass M-test (choosing $l > 1$), we see that $\sum a_k e^{ikx}$ converges uniformly to a continuous function u. The same argument also shows that $\sum (ik)^l a_k e^{ikx}$ converges uniformly to u_l. It is easy to see (fundamental theorem of calculus and interchange of summation and integration) that $u_l = u^{(l)}$. □