ATM SCHOOL 2018 - SUBHARMONIC AND PLURISUBHARMONIC FUNCTIONS (LIBERALLY COPIED FROM KRANTZ'S BOOK, AND D. VAROLIN'S NOTES)

1. Recap

(1) Proved a useful equivalent characterisation of SH functions by means of the local μ-SMVP.
(2) As a corollary we proved several things like $u_1 + u_2$ is SH if u_1, u_2 are so.
(3) Almost finished the proof of u is SH iff $\Delta u \geq 0$ in the sense of distributions.

2. The Perron method

Now we describe Perron's method to solve the Dirichlet problem. Assume that $\Omega \subset \mathbb{C}$ is smoothly bounded and $f : \partial\Omega \to \mathbb{R}$ is continuous. Let S be the set of all continuous subharmonic functions less than f on the boundary. It is non-empty because the constant function $u_0(x) = \min f$ belongs to S.

Let $u(x) = \sup_{g\in S} g(x)$. Of course u is well-defined by subharmonicity of g. The claim is that u is harmonic and solves the Dirichlet problem. (For the latter, it is crucial that the boundary be "nice" for this. Being smooth suffices.)

Firstly, given $u_0(x)$, we can construct "bigger" functions that are harmonic on at least small discs in Ω. Indeed, the Dirichlet problem on a disc can be solved explicitly : If h is continuous on the boundary of the ball $D(y,r) \subset \mathbb{R}^n$, then here is the explicit solution (the Poisson integral).

$$v_h(x) = \frac{r^2 - |x-y|^2}{rArea_{n-1}} \int_{\partial D(y,r)} \frac{h(z)}{|z-(x-y)|^n} d\sigma_z. \tag{2.1}$$

Exercise : Prove that indeed $v(x)$ is a solution to the Dirichlet problem and (using the strong maximum principle) that it is unique.

Now the following function (the harmonic replacement) is continuous and subharmonic if $u(x)$ is subharmonic and continuous : $w(x) = u(x)$ *on* $\Omega - B_r(y)$ and $w(x) = v_u(x)$ *on* $B_r(y)$. Indeed, this can be using the definition.

Consider any sequence of subharmonic functions (bounded below) less than f on the boundary $u_n(y)$ increasing to $u(y)$ (obtained replacing an arbitrary sequence $u_n \in S$ by $\max\{u_1, u_2, \ldots, u_0\}$ if necessary). Take any ball $B_R(y)$ in Ω. Now the harmonic replacements w_n on u_n (on the ball) are also increasing by the maximum principle and converge to some v.

We claim that v is harmonic on $B_{R/2}(y)$. One way is the following : $w_k - w_l$ is harmonic on $B_R(y)$. So the Harnack inequality implies that $|w_k(z) - w_l(z)| \leq C|w_k(x) - w_l(x)|$ for all $z \in B_{R/2}(y)$ and all $x \in \partial B_{R/2}(y)$. Thus $w_k \to v$ uniformly. The uniform limit of harmonic functions is harmonic (the MVP is satisfied in the limit). Note that $u(y) = v(y)$.

Now we prove that $v = u$ on $B_{R/2}(y)$. Of course $v \leq u$. Suppose at some point y_0, $u(y_0) > v(y_0)$. Then by definition of u, there is a $q \in S$ such that $u(y_0) > q(y_0) > v(y_0)$. Now using $\tilde{w}_k = v_{\max(w_k,q)}$ we get a sequence uniformly converging to a harmonic function $\tilde{q}$ on $B_{R/2}$ satisfying $v \leq \tilde{q} \leq u$ implying that $v(y) = q(y) = \tilde{q}(y) = u(y)$. This is a contradiction by the MVP.

We are not done yet. We still have to show that u satisfies the boundary condition we need. For this we need the notion of a "barrier" function.

Definition 2.1. Let $y \in \partial\Omega$. A function $\beta \in C^0(\bar{\Omega})$ is called a barrier at y if $\beta(y) = 0$ and $\beta(x) > 0$ for $x \neq y$ and β is superharmonic in Ω (i.e., $-\beta$ is subharmonic).
A point $y \in \partial\Omega$ is called regular if there exists a barrier function at y.

Exercise : Prove that if the Dirichlet problem can be solved for every continuous boundary datum, then every point is regular.

Remark 2.2. If we have a local barrier, then we can construct a global one.

Lemma 2.3. *If y is a regular boundary point, then* $\lim_{x\to y} u(x) = f(y)$.

Proof. Firstly, the $\limsup$ of the above limit is $\leq f(y)$. Consider $v_1(x) = f(y) + \epsilon + A\beta(x)$ and $v_2(x) = f(y) - \epsilon - A\beta(x)$ where $\epsilon > 0$ is arbitrary and fixed, $|f(x) - f(y)| \leq \epsilon$ for $|x - y| \leq \delta$, and A is chosen so that $A\beta(x) \geq 2\sup_{x\in\bar{\Omega}} |u|$ for all $|x - y| \geq \delta$. Then by the maximum principle $v_2 \leq u \leq v_1$ because v_1 is superharmonic and v_2 is subharmonic. □

Now we come to Poincaré 's exterior disc condition for regularity of a point.

Theorem 2.4. *Suppose that $B_r(y) \cap \Omega = \phi$ and $\overline{B_r(y)} \cap \partial\Omega = \{z\}$ with $r > 0$. Then z is a regular point.*

Proof. For $n \geq 3$, take $\beta(x) = -\frac{1}{|x-y|^{n-2}} + \frac{1}{r^{n-2}}$ and for $n = 2$, take $\beta(x) = \ln|x - y| - \ln r$. □

Exercise : Prove that for smooth domains, the exterior disc condition is satisfied for every point. Thus the Dirichlet problem can be solved.

3. The Riemann mapping theorem

Suppose Ω is a smoothly bounded simply connected domain in $\mathbb{C}$. For a disc whose boundary is ground, if one puts a negative charge at the centre, the equipotential lines are circles with the electric field lines being perpendicular. If one takes Ω, grounds its boundary and puts the same charge somewhere inside, one can hope to map the equipotential lines to circles and the electric field lines to radial lines thus proving the Riemann mapping theorem in this important special case. This is how Riemann proved it.

Indeed, let $p \in \Omega$ and $u(x)$ be the smooth solution of $\Delta u = 0$, $u(z) = \ln|z - p|$ *on* $\partial\Omega$. Then $G(z) = \ln|z - p| - u$ is harmonic on $\Omega - p$, 0 on $\partial\Omega$ and has a unit "charge" put at p. Since Ω is simply connected, $v(z) = \displaystyle\int_p^z (-u_y, u_x).d\vec{l}$ is well-defined, smooth, and is a harmonic conjugate of u. Thus $f(z) = e^{u+iv}(z - p)$ is locally of the form $e^{G+i(v+Arg(z-p))}$ where $v + Arg(z - p)$ is locally the harmonic conjugate of G. f is holomorphic and takes only p to 0. By the maximum principle $G \leq 0$ inside Ω. Thus $|f| \leq 1$ and $f : \Omega \to \mathbb{D}$. Moreover, $f'(p) > 0$. (So the multiplicity of the root is 1.) As $z_n \in \Omega \to \partial\Omega$, $|f(z_n)| \to 1$. In other words, f is proper. So $f^{-1}(a)$ for any point in $\mathbb{D}$ is a finite collection of points (which is 1 for $a = 0$).

Unfortunately, some manifold theory is needed at this point. The degree of f is 1 and hence f is 1-1 and onto. This means that f is a biholomorphism.

4. Plurisubharmonic functions, properties

This material was not covered by me on 22 June but I am posting the notes nonetheless.

Just as harmonic functions are real parts of holomorphic functions in $\mathbb{C}$, the real parts of such things in $\mathbb{C}^n$ are pluriharmonic, i.e., $\frac{\partial^2}{\partial z^i \partial \bar{z}^j} u = 0 \ \forall \ i, j$. (This is an easy calculation to verify.)

Exercise : Suppose $f = u + iv : \mathbb{C}^n \to \mathbb{C}$ is a C^2 function such that it is holomorphic in each of the variables separately. Then prove that u, v are pluriharmonic. Also, prove that pluriharmonic functions are harmonic.

Remark 4.1. It turns out (using the Poincarè lemma on $\bar{\partial} f - \partial f$) that the converse is also true locally (more generally, on a simply connected domain).

Just as subharmonic functions prove to be useful in $\mathbb{C}$, we define plurisubharmonic functions.

Definition 4.2. An u.s.c function $f : \Omega \subset \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ (where as always Ω is open and connected)

is called plurisubharmonic (psh) iff

(1) It is identically $-\infty$, and
(2) for each affine complex line in $\mathbb{C}^n$, $u|_{L\cap\Omega}$ is either subharmonic or $-\infty$ identically.

For example, if f is a holomorphic, non-zero function, then $\ln |f|^2$ is psh. Indeed, $\ln |f(a + bz)|^2$ is sh or $-\infty$ in z by the above.

Exercise : Prove that C^2 function u is psh iff $\frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} w^i \bar{w}^j \geq 0 \ \forall \ \vec{w} \in \mathbb{C}^n$.

Exercise : Prove that the SMVP for a psh function u is satisfied for polydiscs and balls. (Hint : You may have to use Tonelli's theorem.) Conclude that psh functions are locally integrable. Now if $\chi(z^1, z^2, \ldots, z^n) = \chi(|z^1|, \ldots, |z^n|)$ is a smooth compactly supported function with non-zero integral, then prove the μ-SMVP for polydiscs with the measure χdV.

The same proofs as before carry over to prove the following result :

Suppose u is psh.

(1) If $c > 0$, then cu is psh.
(2) If $u = \sup_\alpha u_\alpha(x)$ is finite and u.s.c, and u_α are psh, then so is u.
(3) If $u_1 \geq u_2 \geq \ldots$ and $u = \lim u_i \neq -\infty$ identically, then u is psh if u_i are psh.

We need regularisation for later use. Suppose $\chi(z^1, \ldots, z^n) = \chi(|z^1|, \ldots, |z^n|) \geq 0 : \mathbb{D}^n \to [0, \infty)$ is smooth, compactly supported, and satisfies $\displaystyle\int \chi dV = 1$. If u is in $psh(\Omega)$ then for $\epsilon > 0$, $u_\epsilon(z) = \displaystyle\int_{\mathbb{C}^n} u(z - \epsilon\zeta)\chi(\zeta) d\zeta = \int u(y) \frac{\chi((z-y)/\epsilon)}{\epsilon^{2n}} dy$ is smooth and psh on Ω_ϵ. It is easy to see that u_ϵ is psh. The μ-SMVP shows that $u_\epsilon \geq u$. If u was smooth, then u_ϵ would have decreased to u (by the same argument as before for sh functions). By a double convolution trick we can prove the same thing now. So u can be approximated by a decreasing sequence of smooth PSH functions.

By applying Jensen's inequality we get

Theorem 4.3. *If $u_1, \ldots, u_n$ are psh, and $\phi : \mathbb{R}^n \to \mathbb{R}$ is convex, and separately increasing in each variable, then $\phi(u_1, u_2, \ldots, u_n)$ is psh.*
In particular, $u_1 + u_2 + \ldots + u_n$, $\max(u_1, u_2, \ldots, u_n)$, and $\ln(e^{u_1} + e^{u_2} \ldots)$ are psh.

Proof. We just need to verify the convexity of $(x^1, \ldots, x^n) \to \sum x^i$, $\max(x^i)$, $F(x) = \ln(\sum e^{x^i})$. The first two are obvious whereas we shall compute D^2F for the last one and prove that it is positive-definite. Indeed,

$$D^2F = (\sum e^{x_k})^{-2} \begin{pmatrix} \sum_{k\neq 1} e^{x_1+x_k} & -e^{x_1+x_2} & \ldots & -e^{x_1+x_n} \\ -e^{x_1+x_2} & \sum_{k\neq 2} e^{x_2+x_k} & \ldots - e^{x_2+x_n} & \\ \vdots & \vdots & \ldots & \vdots \end{pmatrix} \tag{4.1}$$

Now $D^2F(v,v) = \sum_{i<j}(v_i - v_j)^2 e^{x_i+x_j}$. □

We have a characterisation in terms of distributional derivatives :

Theorem 4.4. *If an u.s.c function u is psh, then for all smooth compactly supported $\phi \geq 0$, $\displaystyle\int u \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j} w^i \bar{w}^j dV \geq 0$ for all $\vec{w}$.*
Conversely, if a locally integrable f satisfies $\displaystyle\int f \frac{\partial^2 \phi}{\partial z^i \partial \bar{z}^j} w^i \bar{w}^j dV \geq 0$ for all $\vec{w}$, ϕ as above, then it agrees almost everywhere with a psh function.

Proof. Note that if u is psh, then u_ϵ is psh and decrease to u. Now for smooth psh functions the inequality is clear. Since $u_\epsilon \to u$ in L^1_{loc}, we are done.

f_ϵ is smooth and satisfies $\partial\bar{\partial} f_\epsilon \geq 0$ and hence so in the sense of distributions. If we prove that f_ϵ decreases to f, then indeed f is psh. This proof is by the double convolution trick as before. □

Here is an important lemma (which is used to define the notion of psh function on a manifold).

Lemma 4.5. *Let $\Omega_1 \subset \mathbb{C}^m$ and $\Omega_2 \subset \mathbb{C}^n$. If $F : \Omega_1 \to \Omega_2$ is holomorphic, then $F^* psh(\Omega_2) \subset psh(\Omega_1)$.*

Proof. Note that $\partial\bar{\partial} F^* u_\epsilon = F^* \partial\bar{\partial} u_\epsilon \geq 0$. Now $F^* u_\epsilon = u_\epsilon \circ F \to u \circ F = F^* u$ pointwise. Since the decreasing limit of a sequence of psh functions is psh, we are done. □

Exercise : Prove that $\ln(|f_1|^2 + |f_2|^2 + \ldots + |f_k|^2 + \epsilon^2)$ is psh whenever f_i are holomorphic for all $\epsilon \in \mathbb{R}$. For $\epsilon = 0$ assume that the function is not identically $-\infty$.

Lastly, here is an important definition :

Definition 4.6. A C^2 function u is is said to be strictly psh if $h_{i\bar{j}} = \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j}$ is a positive-definite matrix.

It is easier to see that $\ln(1 + \sum_{i=1}^n |z^i|^2)$ is strictly psh. The corresponding Hermitian matrix h_{ij} is called the Fubini-Study metric.

5. The complex Monge-Ampère equation

Just as $\Delta \ln(|z|^2) = \delta_0$ (it is harmonic everywhere but 0 and integrates to a constant in any neighbourhood of 0), what should the analogous statement be for $G = \ln(\sum_{i=1}^n |z^i|^2)$ in higher dimensions ? This function is definitely psh. But if we compute $\partial\bar{\partial} G$, we do not get 0 ! However, it is not too hard to see that the determinant of this matrix is 0 a.e. If we integrate the determinant over small neighbourhoods of the origin, we will get a constant. So this is sort of a "Green" function for the equation $\det(\frac{\partial^2 u}{\partial z^i \bar{\partial} z^j}) = 0$. This is an example of the complex Monge-Ampére equation $\det(\frac{\partial^2 u}{\partial z^i \bar{\partial} z^j}) = f$.

Unfortunately, this equation (being fully nonlinear) is very difficult to solve and needs some restrictions. In particular, putting f to 0 is a bad idea in general. For instance, here is a non-smooth solution of the (homogeneous) complex Monge-Ampère equation : $u(z_1, z_2) = h(z_1)$ where h is any C^2 but not smooth function.

Usually one requires $f > 0$ to get smooth solutions. But even that is not good enough. The boundary needs to be more than just smooth. (It has to be strongly pseudoconvex.) In fact, even for the ball it is not clear as to how to solve the Dirichlet problem. There is an analogue of the Perron method that produces a solution. The other approach is through fully nonlinear PDE due to Caffarelli-Kohn-Nirenberg-Spruck.

The complex MA equation plays an important role in not just the study of domains in $\mathbb{C}^n$ but also compact complex manifolds. Its global version is called the Calabi conjecture and was solved by Yau.