MA 200 - Lecture 17

1 Recap

1. Improper integrals, their properties, criteria for integrability, and relationship with usual integrals.

2 Improper integrals

Theorem 1. *Let $A \subset \mathbb{R}^n$ be open and $f : A \to \mathbb{R}$ be continuous. Let $U_1 \subset U_2 \ldots$ be open sets whose union is A. Then the improper integral exists iff $\int_{U_N} |f|$ is an existent bounded sequence. Its limit is the improper integral.*

Proof. Assume $f \geq 0$ as usual. If the improper integral exists, by monotonicity $\int_{U_N} |f|$ exists and is bounded. Moreover, its limit is $\leq$ the improper integral.
If the sequence exists and is bounded, then let D be a compact rectifiable subset of A. It can be covered by one of the U_i. By monotonicity, we are done. $\square$

Here are a couple of examples.

1. Let $A = \{x > 1 \; and \; y > 1$ and $f(x, y) = \frac{1}{x^2y^2}$. Now choose $U_N = (1, N) \times (1, N)$. Note that U_N is rectifiable and f being continuous on U_N is R.I and by Fubini and FTC, $\int_N f = (1 - 1/N)^2$ whose limit is 1.
2. Let $B = (0, 1)^2$ and f as before. Choose $U_N = (1/N, 1) \times (1/N, 1)$. As before we can integrate to see that the sequence is not bounded and hence the improper integral does not exist.

3 Partitions-of-unity

Our aim is to eventually prove Green's theorem, a special case of which is, $\int_C (Pdx + Qdy) = \int_A (Q_x - P_y) dA$ where $A = \{f < 0\}$ where $f = 0 = C$ is a regular level set, and C is a smooth regular simple closed parametrised path, P, Q are smooth in a neighbourhood of $\bar{A}$. Note that it is trivial to prove this theorem when A is a rectangle (and in fact, even for a triangle). In UM 102, we mumbled something about breaking A into a bunch of triangles and rectangles, adding stuff up, and taking a limit. Instead of this strategy, we shall still split up $\bar{A}$ into pieces, but rather than adding integrals, we decompose f as a sum of compactly supported smooth functions. The pieces we shall split $\bar{A}$ into are A itself and a cover of C by coordinate charts where a neighbourhood of C looks like the upper half-plane. Then we need to know that the integrals change correctly under change of coordinates and so on. So we need two important ingredients now:

1. $1 = \sum_i \rho_i$ where ρ_i are smooth compactly supported functions such that their supports lie in specified open sets, and every point has a neighbourhood intersecting only finitely many supports (So that this is a finite sum). Such functions form a 'partition-of-unity'.
2. A formula that tells us how integrals change when we change variables. It will be nice to have this formula for improper integrals (to be able to integrate the Gaussian for instance).

Later on, we shall also need to generalise Green to higher dimensions. To that end, we need to generalise $\bar{A}$ to oriented manifolds-with-boundary, generalise the notion of a "cross product" and that of the "curl". We shall define differential forms (the integrands), the wedge product (cross product), and the exterior derivative (curl).
With this motivation, we proceed to partitions-of-unity. Before that recall that we had already produced smooth non-negative functions with compact support in $\mathbb{R}$ with support in any arbitrary closed interval. By multiplying $g(x_1, \ldots, x_n) = f_1(x_1)f_2(x_2)\ldots$, we get a smooth function g with compact support in a given rectangle Q such that $g > 0$ on the interior of Q.

Theorem 2. *Let $A = \cup_{j \in J} A_j$ where A_j are open subsets of $\mathbb{R}^n$ (and J need not be countable). There exists a sequence $\rho_i : \mathbb{R}^n \to \mathbb{R}$ of smooth non-negative functions with compact supports $S_i \subset A_{j_i}$ such that $\sum_i \rho_i = 1$ and every point in A has a neighbourhood that intersects only finitely many S_i.*

Such a collection ϕ_i is said to be a partition-of-unity subordinate to/dominated by the open cover $\{A_j \; j \in J\}$.

Proof. To produce such functions, we first produce a countable collection of rectangles $Q_i \subset A_{j_i}$ such that $A = \cup_i Int(Q_i)$, and every point of A has a neighbourhood intersecting only finitely many of the Q_i. Assuming the existence of such Q_i, we can complete the proof quickly: Consider the smooth functions ψ_i with support in Q_i and $\psi_i > 0$ on $Int(Q_i)$. The key point is that $\lambda(x) = \sum_i \psi_i(x)$ is actually a *finite* sum in a neighbourhood of x and is hence a smooth function on A. Since Q_i cover A, $\lambda(x) > 0$ on A. Now define $\rho_i = \frac{\psi_i(x)}{\lambda(x)}$ to be done.
Now we produce the Q_i. Note that even if we did not have the "local finiteness" condition, it is not easy to produce them. To meet local finiteness, we take a compact exhaustion C_N of A. Let $B_N = C_N - Int(C_{N-1})$. Then B_N are compact sets such that $B_N \cap C_{N-2} = \phi$ and $\cup_N B_N = A$. For each $x \in B_N$, consider a closed cube $R_x \subset A_{i_x}$ and disjoint from C_{N-2}. We can choose finitely many such cubes $R_{N1}, R_{N2}, \ldots, R_{Nk_N}$ whose interiors cover B_N. The countable collection R_{ij} of cubes is such that their interiors cover A (because each point of A is in one of the B_N). Given any point $a \in A$, it lies

in $Int(C_N)$ for some N. Thus it is disjoint from $C_{N+2}, \ldots$. Hence $Int(C_N)$ can intersect only finitely many of the cubes C_{ij} where $i \leq N+1$. $\square$

Now the improper Riemann integral can be written (but not calculated) in a nice way using partitions-of-unity:

Theorem 3. *Let $A \subset \mathbb{R}^n$ be open and $f : A \to \mathbb{R}$ be continuous. Let ϕ_i be a (compactly supported) partition-of-unity on A. Then the improper integral of f over A exists iff the series $\sum_i \int_A \phi_i |f|$ converges and in this case, $Improper\ \int_A f = \sum_i Improper \int_A \phi_i f$.*

Proof. The key point is that if a continuous function g has support in a compact subset C, then $\int_A g = \int_C g$: The fact that the integral over C exists follow from the criterion of integrability (because the subset of the boundary of C where g does not tend to 0 is empty). Let C_N be a compact rectifiable exhaustion of A. Then since we need finitely many interiors of C_N to cover C, $C \subset C_K$ for some K. By the same reasoning, $\int_{C_K} g$ exists and since $g_{C_k} = g_C$, the integrals are equal. Applying this reasoning to $|g|$, we see that $\int_A g$ exists and equals $\int_C g$.
Since $Improper\ \int_A f$ exists iff that of $|f|$ does, assuming we have proven the result for $f \geq 0$, we see that the improper integral exists iff $\sum_i \int \rho_i |f|$ does and since $\int_A f = \int_A f_+ - \int_A f_- = \sum_i \int_A (\phi_i (f_+ - f_-)) = \sum_i \int_A (\phi_i f)$. So WLog we can assume that $f \geq 0$.
Suppose C_N is a rectifiable compact exhaustion of A. Then the improper integral exists iff $\int_{C_N} f \leq E$.
Suppose $\int_{C_N} f \leq E$: $\sum_{i=1}^K \int_A \rho_i f = \int_A (\sum_{i=1}^k \rho_i f) \leq \int_{C_{N_k}} f \leq K$ for some N_k such that C_{N_k} contains the supports of $\rho_1, \ldots, \rho_K$. Thus the series converges. Moreover, this argument also shows that $\sum_{i=1}^K \int_A \rho_i f \leq Improper\ \int_A f$.
Suppose the series converges: Let $D \subset A$ be a compact rectifiable subset. Then there is an M so that for $i > M$, ρ_i vanishes on D (indeed, one can cover D by finitely many open sets such that each intersect only finitely many supports). Thus, $\int_D f = \int_D (\sum_{i=1}^M \rho_i f) = \sum_{i=1}^M \int_D \rho_i f \leq \sum_{i=1}^M \int_{D \cup S_i} \rho_i f = \sum_{i=1}^M \int_A \rho_i f \leq \sum_{i=1}^\infty \int_A \rho_i f$. Thus we see that the improper integral exists and is $\leq$ the sum of the series. $\square$