MA 200 - Lecture 26

1 Recap

1. Defined the exterior derivative.
2. Defined pullbacks and proved a few properties.

2 Pullback

Here are some examples of pullbacks:

1. $\omega = dx \wedge dy$. Let $F(r,\theta) = (r\cos(\theta), r\sin(\theta))$. Then $F^*\omega = F^*dx \wedge F^*dy$. Now $F^*dx(v,w) = dx(DF(v,w)) = dx(\frac{\partial r\cos(\theta)}{\partial r}v + \frac{\partial r\cos(\theta)}{\partial\theta}w, \ldots) = \cos(\theta)v - r\sin(\theta)w = d(r\cos(\theta))(v,w)$. Indeed, more generally, $F^*df = d(f \circ F)$ because $F^*df(v) = df(DFv) = \sum \frac{\partial f}{\partial x_i}\frac{\partial F_i}{\partial x_j}v_j = d(f \circ F)(v)$.

2. If $\gamma(t) = (\cos(t), \sin(t))$, then $\gamma^*dx = d(x \circ \gamma) = d(\cos(t)) = -\sin(t)dt$ as we expect.

3. More generally, $F^*(d\omega) = d(F^*\omega)$: Indeed, $F^*(\sum_i d\omega_I \wedge \epsilon_I) = \sum_i F^*(d\omega_I) \wedge F^*dx_{i_1} \wedge \ldots = \sum_i d(\omega_I \circ F) \wedge dF_{i_1} \ldots$. Now $d(F^*\omega) = d(\sum_I \omega_I \circ F dF_{i_1} \wedge \ldots) = \sum_I dF^*\omega_I \wedge dF_{i_1} \ldots + F^*\omega_I d(dF_{i_1}) \wedge \ldots + \ldots$ but $ddf = 0$ and hence we are done. $\square$

4. Suppose $x_1, \ldots, x_n$ are coordinates in $\mathbb{R}^n$, $(y_1, \ldots, y_k)$ in $\mathbb{R}^k$, $F : U \subset \mathbb{R}^k \to \mathbb{R}^n$ is a smooth map, and I is an increasing multi-index of size k. Then $F^*(\epsilon_I)(e_1, \ldots, e_k) = \epsilon_I(DFe_1, \ldots, DFe_k) = \det(\frac{\partial(F_{i_1}, \ldots)}{\partial(y_1, \ldots, y_k)})$. In other words, $F^*\epsilon_I = \det(\frac{\partial(F_{i_1}, \ldots)}{\partial(y_1, \ldots, y_k)})dy_1 \wedge dy_2 \ldots dy_k$.

3 Integrating top forms in $\mathbb{R}^n$

Let ω be a smooth n-form field in an open subset $U \subset \mathbb{H}^n$ or $\mathbb{R}^n$. Then $\omega = f dx_1 \wedge dx_2 \ldots dx_n$. Define $\int_U \omega = \int_{Int(U)} f$ as an improper integral if it exists.The point of the definition is the following: Suppose $\phi : V \to Int(U)$ is a smooth diffeomorphism (that is, a homeomorphism that is smooth and whose derivative is injective throughout), then by the change of variables formula, $\int_{Int(U)} f = \int_{Int(V)} f \circ \phi |\det(D\phi)|$. If ϕ is orientation-preserving, then $\int_U f = \int_V f \circ \phi \det(D\phi)$. However, recall that $\phi^*(dx_1 \wedge \ldots) = \det(D\phi)dy_1 \wedge \ldots$. Thus, $\int_U \omega = \int_V \phi^*\omega$ provided ϕ is orientation-preserving. Otherwise, $\int_U \omega = -\int_V \phi^*\omega$.

Stokes' theorem in $\mathbb{H}^n$: Let ω be a smooth $n-1$-form field with compact support in $\mathbb{H}^n$. Then $\int_{\mathbb{H}^n} d\omega = (-1)^n \int_{x_n=0} i^*(\omega)$, where $i(x_1, \ldots, x_{n-1}) = (x_1, \ldots, x_{n-1}, 0)$.

Proof. Suppose $\omega = \omega_1 dx_2 \wedge dx_3 \ldots + \omega_2 dx_1 \wedge dx_3 + \ldots$. The integral is (using Fubini's theorem) $\int_0^a \int_{-a}^a \ldots \sum_i (-1)^{i-1} \partial_i \omega_i dx_1 dx_2 \ldots dx_n$. Using the fundamental theorem of calculus, we see that $\int_{-a}^a \partial_i \omega_i dx_i = 0$ whenever $1 \leq i \leq n-1$ because ω has compact support. For $i = n$, we get $\int_{-a}^a \ldots (-1)^n \omega_n (x_1, \ldots, x_{n-1}, 0) dx_1 \ldots$. $\square$

The same proof shows that if ω has compact support in $\mathbb{R}^n$, then $\int_{\mathbb{R}^n} d\omega = 0$.

4 Differential forms on manifolds (with or without boundary)

Let $M \subset \mathbb{R}^n$ be a smooth k-manifold-with-boundary. A smooth l-form field on M is a map $\omega : M \to \cup_p \Lambda^l(T_p M)$ such that $\omega(p) \in \Lambda^k(T_p M)$ and for any smooth parametrisation α, the form field $\tilde{\omega}(x)(v_1, \ldots, v_l) = \omega_{\alpha(x)}(D\alpha v_1, \ldots)$, i.e., $\alpha^* \omega$ is smooth. Clearly, if ω is a smooth l-form field in an open neighbourhood U of M in $\mathbb{R}^n$, then when restricted to M, it is a smooth form field (by the chain rule). It can be proven that every smooth l-form field can be extended to a smooth l-form field in a neighbourhood of M. Here are examples of form-fields:

1. The 1-form field $\omega = \frac{xdy - ydx}{x^2+y^2}$ on the unit circle. If we choose the parametrisation of a part of the unit circle given by $\alpha(t) = (\cos(t), \sin(t))$, then $\alpha^*(\omega) = dt$.

2. The 2-form field $\omega = z^2 dx \wedge dy + x^2 dy \wedge dz$ on the unit upper hemisphere. If we parametrise a part of it using $(x, y, \sqrt{1-x^2-y^2})$, then $\alpha^* \omega = (1 - x^2 - y^2 + \frac{x^3}{\sqrt{1-x^2-y^2}}) dx \wedge dy$.