MA 200 - Lecture 13

1 Recap

1. Taylor in multivariable calculus.
2. Second derivative test.
3. Started partitions of rectangles.

2 Integration in more than one-variable

Lemma 2.1. *Let P be a partition of a rectangle Q and $f : Q \to \mathbb{R}$ be a bounded function. If P' is a refinement of P, then $L(f, P) \leq L(f, P')$ and $U(f, P') \leq U(f, P)$.*

Proof. Let k be the number of points in $P'_1 - P_1$ plus those in $P'_2 - P_2$ plus so on. We induct on k. In fact, we claim that $k = 1$ is enough. Indeed, if it is true for $1, 2 \ldots, k-1$, then replace P with the partition obtained by adding the $k-1$ points and then apply the $k = 1$ case.
For $k = 1$: Suppose the additional point b is in the i^{th} component interval and in the sub-interval $[t_{ij}, t_{ij+1}]$. Then the rectangles $R = I_1 \ldots I_{i-1} \times [t_{ij}, t_{ij+1}] \times I_{i+1} \ldots$ are replaced by $R_1 = I_1 \ldots I_{i-1} \times [t_{ij}, a] \times I_{i+1} \ldots$ union $R_2 = I_1 \ldots I_{i-1} \times [a, t_{ij+1}] \times I_{i+1} \ldots$. The infimum increases if the size of the size is reduced (why?) and the supremum decreases. Hence $m_{R_1}, m_{R_2} \geq m_R$, $M_{R_1}, M_{R_2} \leq M_R$ and since $v(R) = v(R_1) + v(R_2)$, $m_R v(R) \leq m_{R_1} v(R_1) + m_{R_2} v(R_2)$ and likewise for M_R. Thus we are done. $\square$

Moreover, if P, P' are any two partitions, then $L(f, P) \leq U(f, P')$ (and as a consequence, the lower R.I is $\leq$ the upper one): Indeed, let C be their common refinement. Then $L(f, P) \leq L(f, C) \leq U(f, C) \leq U(f, P')$.
Def: Let $f : Q \to \mathbb{R}$ be a bounded function and $Q \subset \mathbb{R}^n$ be a closed rectangle. Then the lower Riemann integral $\underline{\int_Q} f dV$ is the supremum over all partitions of $L(P, f)$ and the upper Riemann integral $\overline{\int_Q} f dV$ is the infimum over all partitions of $U(P, f)$. These numbers always exist. f is said to be Riemann integrable with integral $\int_Q f dV$ if these numbers are equal and $\int_Q f dV = \underline{\int_Q} f dV = \overline{\int_Q} f dV$.
Example: A constant function is Riemann integrable with integral $cv(Q)$. Indeed, consider the trivial partition to conclude that the upper and lower Riemann sums and hence the integrals are equal and that too to $cv(Q)$. Now if P is any other partition, since $m_R = M_R = c$, we see that $v(Q) = \sum_R v(R)$ (an interesting identity).

Example: A piecewise-constant function on Q is a partition P_0 together with constants $c_{i_1 i_2 \ldots i_n}$ for each open subrectangle and arbitrary values on the boundaries. Piecewise-constant functions are R.I with integral $\sum_I c_I v(R_I)$ (where I is a multi-index). Indeed, given any partition P, consider the common refinement of P_0, P. Consider an even further refinement by adding points on both sides of the points in $(P_0)_i$ with distance $\epsilon > 0$. Now the lower and upper Riemann sums are within $C\epsilon$ of each other and $\sum_I c_I v(R_I)$ (why?) Hence, the upper and lower R.I are within $C\epsilon$ of each other. Since ϵ is arbitrary, we are done.
Non-example: The Dirichlet function $f(x) = 1$ when x is a rational and $f(x) = 0$ when x is irrational is not R.I over $[0, 1]$.

Theorem 1. *Riemann's criterion: A bounded function $f : Q \to \mathbb{R}$ is R.I iff for every $\epsilon > 0$, there is a partition P_ϵ such that $U(P_\epsilon, f) - L(P_\epsilon, f) < \epsilon$.*

The proof is exactly the same as in the $1 - D$ case.

3 Measure zero sets and integrability (from Munkres)

We proved that piecewise-constant functions are integrable. So what went wrong with the Dirichlet function? It isn't just that there were infinitely many discontinuities (you will see an example in your HW). It is that probabilistically speaking, if you throw a dart at them, you are guaranteed to hit a discontinuity (since probability is presumably vaguely related to length, we already see a problem). Lebesgue proved a criterion that decided Riemann integrability (based on this probabilistic intuition).
Instead of probabilities, let us try to define when a set has zero volume:
Def: Let $A \subset \mathbb{R}^n$. It is said to have measure zero in $\mathbb{R}^n$ if for every $\epsilon > 0$ there is a cover of A by countably many closed rectangles $Q_1, \ldots,$ such that $\sum_i v(Q_i) < \epsilon$. (This is abbreviated as "the total volume of this cover is less than ϵ".)
What if A is a rectangle itself? At least if we cover a closed rectangle Q by finitely many rectangles $Q_1, \ldots, Q_k$, then is $v(Q) \leq \sum_{i=1}^{k} v(Q_i)$? Thankfully yes: Choose a large rectangle Q' containing $Q_1, \ldots, Q_k$. The end points of $Q, Q_1, \ldots, Q_k$ (and those of Q') form a partition of Q'. In particular, the intersection of this partition with Q or Q_1 or ... is a partition of each of them. Thus each of these rectangles is a union of some subrectangles from the bigger partition. We conclude (using the previous results) that $v(Q) = \sum_{R \subset Q} v(R)$. Now each such R is in at least one of the Q_i (because the Q_i form a cover). Thus, $\sum_{R \subset Q} v(R) \leq \sum_{i=1}^{k} \sum_{R \subset Q_i} v(R)$ (finite summations can be done in any order by induction). Now again, $\sum_{R \subset Q_i} v(R) = v(Q_i)$. Hence we are done.
Here is an example of a measure zero set: The rationals in $[0, 1]$ have measure zero: Indeed, they are countable. So enumerate them as $a_1, a_2, \ldots$. Now consider the cover $[a_i - \frac{\epsilon}{2^i}, a_i + \frac{\epsilon}{2^i}]$. The total volume is less than ϵ. In fact, this argument works for any countable subset of $\mathbb{R}$. Even more generally, suppose A is a countable subset of $\mathbb{R}^n$. Then enumerate it as before and consider the cover $[a_{i1} - \frac{\epsilon}{2^i}, a_{i1} + \frac{\epsilon}{2^i}] \times [a_{i2} - \frac{\epsilon}{2^i}, a_{i2} + \frac{\epsilon}{2^i}] \ldots$.
The above example indicates that we genuinely need countably many sets and cannot do with finitely many. Indeed, rationals in $[0, 1]$ have measure zero. If we could cover them by finitely many rectangles whose total volume is $< \frac{1}{2}$, then by density of

rationals, $[0, 1]$ is covered by these rectangles but its volume is 1.
Here are a few properties:

1. If $B \subset A$ and A has measure zero, then so does B (almost by definition).

2. Let $A = \cup_i A_i$ where A_i have measure zero. Then so does A: Indeed, cover A_i by rectangles R_{ij} whose total volume is less than $\frac{\epsilon}{2^i}$. Then the R_{ij} cover all of A. Now $\sum_i \sum_j v(R_{ij}) < \sum_i \frac{\epsilon}{2^i} < \epsilon$. Enumerate the rectangles R_{ij} so that $\sum_n v(Q_n) = \sum_i \sum_j v(R_{ij})$ (but it actually does not matter what order you use because this series is absolutely summable).

3. The closed rectangles in the definition of measure zero can be replaced by their interiors (i.e., open ones): Of course, if A is covered by open rectangles whose total volume is less than ϵ, then their closures also cover A and hence we are done. Conversely, if A is covered by closed rectangles whose total volume is less than $\frac{\epsilon}{10}$, then enlarge the i^{th} closed rectangle by scaling it by a factor of 2. The total volume is still less than ϵ.