MA 200 - Lecture 27

1 Recap

1. Examples of pullbacks.
2. Integration of top forms on $\mathbb{H}^n$ and Stokes theorem.
3. Forms on manifolds.

2 Differential forms on manifolds (with or without boundary)

It can be proven that every smooth l-form field can be extended to a smooth l-form field in a neighbourhood of M. The proof is not trivial but here it is:
Indeed, if ω is a smooth l-form field on M, then suppose $\alpha_i : U_i \to \alpha_i(U_i) = V_i \subset M$ be coordinate parametrisations such that the open sets (in the topology on M) V_i cover all of M. If $p \in V_i$, then by the injective derivative theorem, there exists a local diffeomorphism ϕ_p from an open neighbourhood W_p of p (in $\mathbb{R}^n$) to another open neighbourhood $\phi_p(W_p)$ (in $\mathbb{R}^n$) such that $\phi \circ \alpha_i(x_1, \ldots, x_k) = (x_1, \ldots, x_k, 0, 0 \ldots)$. Now consider the smooth l-form field $\eta_p = ((\phi_p)^{-1})^*(\omega)$ can be written as a linear combination of wedge products of l of the dx_i (where $1 \leq i \leq k$) with coefficients being smooth functions of $x_1, \ldots, x_k$. Now η_p is actually defined on $\phi_p(W_p)$ by trivially extending the coefficients (to independent of the other coordinates). Thus $\tilde{\eta}_p = \phi_p^* \eta_p$ is a smooth l-form field on W_p that extends ω. Now choose a partition-of-unity ψ_p subordinate to the W_p. The smooth l-form field $\sum_p \psi_p \tilde{\eta}_p$ is defined on a neighbourhood of M and extends ω.

The next step is to define the exterior derivative. Since we are assuming that smooth form fields are actually restrictions of those appearing from an open neighbourhood, we simply take the restriction of $d\omega$. We have the following result that directly follows from the properties of the usual exterior derivative.

Lemma 2.1. *Let M be a smooth k-manifold-with-boundary. For every coordinate parametrisation α, $d(\alpha^*\omega) = \alpha^*(d\omega)$ where $d(\alpha^*\omega)$ is the exterior derivative in $\mathbb{R}^k$ and $d\omega$ is well-defined. (Exercise)*

We now define integration of top forms on manifolds-with-boundary. Let ω be a smooth k-form field on an oriented smooth k-manifold-with-boundary M. Assume that ω is compactly supported in a smooth orientation-compatible coordinate

parametrisation $\alpha : U \to M$. Then we define $\int_M \omega := \int_U \alpha^*\omega$. This definition is well-defined because suppose $\beta : V \to M$ is another orientation-compatible coordinate parametrisation containing the support of ω, then since $\beta = \alpha \circ \phi$ where ϕ is an orientation-preserving diffeomorphism (why?), we see that $\int_U \alpha^*\omega = \int_V \phi^*(\alpha^*\omega) = \int_V (\alpha \circ \phi)^*\omega = \int_V \beta^*\omega$. This integral is certainly linear in ω.
Suppose ω is a general smooth k-form field on a compact oriented smooth k-manifold-with-boundary M, then cover M by orientation-compatible smooth coordinate parametrisations α_i. Let ρ_i be a partition-of-unity subordinate to the open cover $\cup_i \alpha_i(\tilde{U}_i)$ (where $\tilde{U}_i$ are open subsets of $\mathbb{R}^n$ containing U_i to which α_i extend smoothly). Define $\int_M \omega = \sum_i \int_M \rho_i \omega$.
This definition coincides with the previous one when ω is supported in a coordinate parametrisation: Indeed, $\sum_i \int_M \rho_i \omega = \sum_i \int_U \alpha^*(\rho_i \omega) = \int_U \alpha^*(\sum_i \rho_i \omega) = \int_U \alpha^* \omega$.
Moreover, this definition is independent of the partition-of-unity chosen: Suppose ψ_j is another partition-of-unity. Then $\sum_j \int_M \psi_j \omega = \sum_j \sum_i \int_M \rho_i \psi_j \omega = \sum_i \sum_j \int_M \psi_j \rho_i \omega = \sum_i \int_M \rho_i \omega$.
This integral is also linear in ω.
Of course this definition is painful to work with in practice. However, just as in the case of scalar fields, we can prove that it is enough to cover M-upto-measure-zero by disjoint sets that are images of orientation-compatible coordinate parametrisations and hence evaluate the integrals and add them up (without any partition-of-unity).

3 The generalised Stokes theorem

Theorem 1. *Let $M \subset \mathbb{R}^n$ be a smooth compact oriented k-manifold-with-boundary. Let ω be a smooth $k-1$-form-field on M. Then*

$$\int_M d\omega = \int_{\partial M} \omega$$

if ∂M is endowed with the induced orientation (that is, the restriction if $dim(M)$ is even and opposite otherwise).

Proof. $\int_M (d\omega) = \sum_i \int_M \rho_i d\omega = \sum_i \int_M d(\rho_i \omega) - \int_M \omega \sum_i d(\rho_i) = \sum_i \int_M d(\rho_i \omega)$. Since the RHS is $\sum_i \int_{\partial M} \rho_i \omega$, we can assume WLOG that ω is compactly supported in a coordinate parametrisation. Then $\int_M d\omega = \int_U \alpha^*(d\omega) = \int_U d(\alpha^*\omega)$. If $U \subset \mathbb{R}^k$ is open, then this integral is zero. Moreover, the RHS is trivially zero in this case. If $U \subset \mathbb{H}^k$, then the earlier Stokes theorem shows that this integral is $\int_{x_k=0} (-1)^k \alpha^*\omega$. The induced orientation's definition implies that this integral is $\int_{\partial M} \omega$. □