MA 200 - Lecture 19

1 Recap

1. Stated and proved change of variables.

2 Change of variables and volumes of parametrised manifolds

As an application of change of variables, we see that if $S \subset \mathbb{R}^n$ is rectifiable and $h(x) = Ax$ (where A is a matrix), then $v(h(S)) = |\det(A)|v(S)$. Indeed, if A is invertible, then h is a change of variables and hence $v(h(S)) = \int_{h(S)} 1 = \int_S 1|\det(A)| = v(S)|\det(A)|$. If A is not invertible, then $h(S)$ is contained in a subspace of dimension $< n$ and hence has measure zero.
Def: If $a_1, \ldots, a_k$ are linearly independent vectors in $\mathbb{R}^n$, then the set $x = \sum_i c_i a_i$ where $0 \leq c_i \leq 1$ is called the k-dim parallelopiped formed by $a_1, \ldots, a_k$.
If $k = n$, since $h(x) = Ax$ takes the standard unit square to the parallelopiped, the volume of a parallelopiped is $|\det(A)|$. This $|.|$ makes for an interesting definition: An ordered basis $a_1, \ldots, a_n$ is said to be positively oriented if $\det(A) > 0$ and is negatively oriented if $\det(A) < 0$. Note that if S is a change of basis, then S preserves orientation if $\det(S) > 0$.
Here is another interesting observation: Let O be an orthogonal matrix (note that an orthogonal matrix preserves inner products). Then $v(OS) = v(S)$ because $\det(O) = \pm 1$. This observation leads to a nice definition of the volume of a k-dimensional parallelopiped where $k < n$:

Theorem 1. *There is a unique function $V : \mathbb{R}^n \times \mathbb{R}^n \ldots \to \mathbb{R}_{\geq 0}$ such that*

1. *If O is an orthogonal matrix, then $V(Oa_1, \ldots, Oa_k) = V(a_1, \ldots, a_k)$.*
2. *If $a_1 = (b_1, 0), \ldots, a_k = (b_k, 0) \in \mathbb{R}^k \times \{0\}$, then $V(a_1, \ldots, a_k) = |\det(B)|$.*

V vanishes iff a_i are linearly independent. Moreover, $V(a_1, \ldots, a_k) = \sqrt{\det(A^T A)}$ where A is the $n \times k$ matrix $A = [a_1 \ \ldots]$.

Proof. Define $F(A) = \det(A^T A)$. If O is an orthogonal matrix, then $F(OA) = F(A)$. Moreover, if $a_i = (b_i, 0)$, then $F(A) = \det(B)^2 \geq 0$ with equality iff b_i (and hence $(b_i, 0)$) are linearly dependent. Now given an arbitrary a, there exists an orthogonal matrix such that $Oa = (b, 0)$ (indeed, take the standard basis to an orthonormal basis that restricts to one on the span of a). Now $V(A) = \sqrt{F(A)}$ satisfies the conditions of the theorem and the argument also shows uniqueness. $\square$

Let M be a C^r parametrised manifold-without-boundary of dimension k in $\mathbb{R}^n$, i.e., $M = \alpha(D)$ where $\alpha : D \subset \mathbb{R}^k \to \mathbb{R}^n$ is a C^r coordinate parametrisation. Then the volume of M is defined as $v(M) = \int_D \sqrt{\det(D\alpha^T D\alpha)}$ if it exists (as an improper integral). Justification:

1. If we take an infinitesimal rectangle in $\mathbb{R}^k$ with sides $dx_i e_i$, then $\frac{\partial \vec{\alpha}}{\partial x_i} dx_i e_i$ are the sides of the infinitesimal parallelopiped it gets mapped to. We know that the volume of this parallelopiped is $\sqrt{\det(D\alpha^T D\alpha)}$.

2. It is independent of parametrisation: Suppose Let $\tilde{\alpha} : \tilde{D} \subset \mathbb{R}^k \to \mathbb{R}^n$ be another C^r coordinate parametrisation (homeomorphic to its image and $D\tilde{\alpha}$ has rank k everywhere). Then this is a reparametrisation of α, i.e., the map $\phi = \tilde{\alpha}^{-1} \circ \alpha : D \to \tilde{D}$ is a C^r diffeomorphism (HW). Now $\tilde{\alpha} \circ \phi = \alpha$ and hence $D\alpha_x = D\tilde{\alpha}_{\phi(x)} D\phi_x \Rightarrow \det((D\alpha_x)^T D\alpha_x) = |\det(D\phi_{(x)})|^2 \det((D\tilde{\alpha}_{\phi(x)})^T (D\tilde{\alpha}_{\phi(x)}))$. Hence by the change of variables formula, we are done.

More generally, if we want to calculate the "charge" or "mass" of a parametrised manifold, then if $f : M \to \mathbb{R}$ is a continuous function ("the charge density"), the integral of f is defined to be $\int_M f dV = \int_D f \circ \alpha \sqrt{\det(D\alpha^T D\alpha)}$. The same proof shows reparametrisation invariance.
Examples:

1. Consider $\alpha : (0, 2\pi) \to \mathbb{R}^2$: $\alpha(t) = (\cos(t), \sin(t))$. As we have shown earlier, α is a smooth coordinate parametrisation. The length of the circle-minus-one point is $\int_0^{2\pi} \sqrt{|\alpha'|^2} dt = 2\pi$. (Note that morally, the one point we missed has measure zero and shouldn't contribute to the circumference. Soon we will define the length of the full circle and you will easily see that indeed our expectation is correct.)

2. Suppose we have a parametrised surface, then $\det(D\alpha^T D\alpha) = \|\frac{\partial \alpha}{\partial u}\|^2 \|\frac{\partial \alpha}{\partial v}\|^2 - |\langle \frac{\partial \alpha}{\partial u}, \frac{\partial \alpha}{\partial v} \rangle|^2 = \|\alpha_u \times \alpha_v\|^2$ just as in UM 102. So using this expression we can calculate the area of the image of $\alpha : (0, \pi) \times (0, 2\pi) \to \mathbb{R}^3$ given by $\alpha(\theta, \phi) = (\sin(\theta)\sin(\phi), \sin(\theta)\cos(\phi), \cos(\theta))$ to be 4π as expected.

3. $\alpha : B_0(1) \to \mathbb{R}^3$ given by $\alpha(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$. Note that α is smooth on D but its derivatives are *not* bounded. Thus the integral must be made sense of as an improper integral. Let $U_N = B_{1-1/N}(0)$. The usual integral over U_N exists and equals $\int_{U_N} \sqrt{1 + \frac{x^2}{1-x^2-y^2} + \frac{y^2}{1-x^2-y^2}} = \int_{U_N} \frac{1}{\sqrt{1-x^2-y^2}}$. We can change variables to polar coordinates (because measure zero sets don't make a difference) and then use Fubini to calculate this integral and get something that converges to 2π.

What about general manifolds that cannot be covered by only one coordinate chart? (By the way, every manifold is, up to measure zero, a parametrised one.) Firstly, given any open cover by coordinate parametrisations, i.e., by parametrised open sets $U_i \cap M$ (where U_i are open in $\mathbb{R}^n$), choose a partition-of-unity for $\cup_i U_i$ subordinate to the open cover. Secondly assume that M is compact and cover it with finitely many coordinate parametrisations - one can find finitely many functions forming a partition-of-unity (local finiteness guarantees that actually any partition-of-unity is finite). Now suppose $f : M \to \mathbb{R}$ is a continuous function. Define $\int_M f dV = \sum_i \int_{D_i} \rho_i f dV_{\alpha_i}$ (note that since $\rho_i f$ has compact support, each summand is a usual integral). Before we prove well-definedness, let us generalise manifolds-without-boundary to having boundary too (that is, instead of only considering the open hemisphere, we want to consider the closed hemisphere with its boundary circle too). After all, this sort of a notion is necessary for Green's theorem.