MA 200 - Lecture 20

1 Recap

1. Defined surface areas/volumes of parametrised manifolds-without-boundary.

2 Manifolds-with-boundary

Consider a hemisphere. What does it look like near the boundary circle? Certainly not like an open subset of $\mathbb{R}^2$ but instead like an open subset of $\mathbb{H}^2 = \{y \geq 0\}$. So we define as follows: A subset $M \subset \mathbb{R}^n$ is called a C^r k-dimensional manifold-with-boundary if it can be covered with open subsets that are homeomorphic to an open subset U of either $\mathbb{R}^k$ or $\mathbb{H}^k$ via $\alpha : U \to \mathbb{R}^n$ that is C^r and $D\alpha$ is $1-1$ everywhere.
Recall that a function $f : S \subset \mathbb{R}^k \to \mathbb{R}^n$ is said to be C^r if it can be extended to a C^r function on an open set containing S. This leads to a problem because a function is C^r at a point $p \in S$ if there is a neighbourhood $V_p \subset \mathbb{R}^n$ of p and a C^r function $\tilde{f}_p : V \to \mathbb{R}^n$ such that $\tilde{f}_p = f$ on $V_p \cap S$. So why is a function C^r iff it is C^r at every point? This fact is true and can be proven using partitions-of-unity: Indeed, assume it is C^r at every point. Now take a partition-of-unity subordinate to the cover V_p of S. Now define $\tilde{f} = \sum_i \phi_i \tilde{f}_{p_i}$. Of course $\tilde{f}$ is C^r on the open set $V = \cup_i V_{p_i} \supset S$ and when $q \in S$, $\tilde{f}(q) = \sum_i \phi_i(q) f(q) = f(q)$. Note also that if a function is C^r on an open subset of $\mathbb{H}^k$, then all the partial derivatives on the boundary points are uniquely determined (Why?)
If α, β are two coordinate parametrisations on a manifold-with-boundary, then $\alpha \circ \beta^{-1}$ is a C^r injective map whose derivative is injective on the intersection of the domains: Indeed, it is enough to show that $\beta^{-1} : M \cap \beta(V) \to \mathbb{R}^k$ can be extended to a C^r map on a neighbourhood. Firstly, if $V \subset \mathbb{H}^k$, then by definition, we can extend β to $\tilde{\beta}$ on an open subset $\tilde{V} \subset \mathbb{R}^k$ containing V. Now by the injective derivative theorem, after composing with diffeos, $\phi \circ \tilde{\beta} \circ \psi^{-1}(y) = (y_1, \ldots, y_k, 0, 0 \ldots)$ whose inverse can certainly be extended to an open set. Hence we are done in this case. In the case where $V \subset \mathbb{R}^k$, we apply this argument to β itself (as opposed to $\tilde{\beta}$).
Def: Let $M \subset \mathbb{R}^n$ be a k-dimensional manifold-with-boundary. A point $p \in M$ is said to be *interior* (NOT in the topological sense) if a neighbourhood of it is coordinate parametrised by an open subset of $\mathbb{R}^k$. It is said to be a boundary point otherwise. The set of boundary points (denoted as ∂M) is called the boundary of M.
The following criteria are useful when $p \in V \subset M$ and $\alpha : U \to V$ is a coordinate parametrisation:

1. If $U \subset \mathbb{R}^k$, p is an interior point (by definition).
2. If $U \subset \mathbb{H}^k$ but p is in $\mathbb{H}^k_{x_k>0}$ then p is an interior point (indeed simply shrink U).
3. If $U \subset \mathbb{H}^k$ and p is on $x_k = 0$, then p is a boundary point (indeed, if not, then a neighbourhood of such a point on $\mathbb{H}^k$ is homeomorphic to an open subset of $\mathbb{R}^k$ by means of a C^r map f whose derivative is an isomorphism. But by the IFT, the image of f^{-1} is open in $\mathbb{R}^k$ whereas the original neighbourhood in $\mathbb{H}^k$ isn't).

Finally, if M is a k-dimensional manifold-with-boundary such that $\partial M \neq \phi$ (typically, this condition is understood), then ∂M is a $k-1$-dimensional manifold-without-boundary in $\mathbb{R}^n$: Indeed, cover ∂M by open sets $\alpha_i(U_i) \cap M = V_i \cap M$ that are boundary coordinate parametrisations for M. Now consider the maps $\tilde{\alpha}_i(x_1, \ldots, x_{k-1}) = \alpha_i(x_1, \ldots, x_{k-1}, 0)$ from $U_i \cap \{x_k = 0\}$ to $V_i \cap \partial M$. These are C^r bijective maps and $D\tilde{\alpha}_i(v_1, \ldots, v_{k-1}) = D\alpha_i(v_1, \ldots, v_{k-1}, 0)$ which is 0 iff $v_j = 0 \,\forall\, j$. Moreover, $\tilde{\alpha}_i$ are homeomorphisms to their images because $\tilde{\alpha}_i^{-1}$ are restrictions of the continuous functions α_i^{-1}. Hence these are coordinate parametrisations.
Lastly, here is a theorem that allows us to prove for instance that the unit disc is a manifold-with-boundary. (Another example is the hemisphere (HW) but it does not follow from this theorem.)

Theorem 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be C^r. Let $N = \{x \mid f(x) \leq 0\}$ and let $\nabla f \neq 0$ for every point on $f^{-1}(0) \neq \phi$. Then N is an n-dimensional manifold-with-boundary in $\mathbb{R}^n$ and $\partial N = f^{-1}(0)$.*

Proof. Note that the set $f < 0$ is open (and hence a manifold-without-boundary) in $\mathbb{R}^n$. Let $p \in f^{-1}(0)$. Assume that $\partial_i f \neq 0$. Consider the map $H : \mathbb{R}^n \to \mathbb{R}^n$ given by $H(x) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, -f)$. Now $\det(DH(p)) \neq 0$ and hence H is a local diffeo from $V \to H(V)$ near p. Moreover, $H_n \geq 0$ iff $f \leq 0$. Thus, $\alpha = H^{-1}$ takes the open set $H(V) \cap \mathbb{H}^k$ homeomorphically to its image and is a boundary coordinate parametrisation. Moreover, $H_n = 0$ iff $f = 0$. Thus $f^{-1}(0) = \partial M$. Lastly, the topological boundary of $f < 0$ is $f = 0$ (why?) □

We now define the integral of certain kinds of functions over manifolds with or without boundary:
Let M be a compact manifold with nonempty or without boundary in $\mathbb{R}^n$ of dimension k. Let $f : M \to \mathbb{R}$ be continuous and compactly supported in $\alpha(U)$ where $\alpha : U \to M$ is a coordinate parametrisation. We define $\int_M f dV = \int_{Int(U)} (f \circ \alpha)\sqrt{\det(D\alpha^T D\alpha)}$. Note that this *a priori* improper integral is actually an ordinary Riemann integral (because $f \circ \alpha$ has compact support), we can assume U is bounded WLog, and the integral is defined over $Int(U)$ because the boundary, being possibly a subset of $\mathbb{H}^n$ has measure zero anyway.
The key point is that this integral is well-defined, independent of the coordinate parametrisation chosen: This follows from the change-of-variables formula. Moreover, linearity holds, i.e., $\int(af + bg) = a\int f + b\int g$ by the linearity of the usual integral.
Here is the general definition: Let M be a compact manifold with nonempty or without boundary in $\mathbb{R}^n$ and $f : M \to \mathbb{R}$ be a continuous function. Let ϕ_i be a (finite) partition-of-unity subordinate to a cover by *all* coordinate parametrisations. Then $\int_M f dV := \sum_i \int_M \phi_i f dV$. When $f = 1$, $\int_M 1 dV$ is called the surface area/volume of M.

1. If f has support in a coordinate patch, this definition coincides with the earlier one: $\sum_i \int_M \phi_i f dV = \sum_i \int_{Int(U)} (\phi_i f) \circ \alpha \sqrt{\det(D\alpha^T D\alpha)} = \int_{Int(U)} \sum_i \phi_i \circ \alpha f \circ \alpha \sqrt{\det(D\alpha^T D\alpha)}$ and we are done.

2. It is independent of the partition-of-unity: If ψ_j is another partition-of-unity subordinate to another cover, then $\sum_j \int_M \psi_j f dV = \sum_j \int_M \sum_i \phi_i (\psi_j f) dV = \sum_j \sum_i \int_M \phi_i \psi_j f dV$ (by linearity) and by linearity again, this equals $\sum_i \int_M \sum_j \phi_i \psi_j f dV = \sum_i \int_M \phi_i f dV$.

Linearity of this general definition is also easy to prove.
Now of course this definition is impossible to work with practically speaking. To this end, we first define the notion of measure zero on a manifold: Let $M \subset \mathbb{R}^n$ be a C^r compact manifold with or without boundary. A subset D is said to have measure zero in M if it can be covered by (the images of) countably many coordinate parametrisations $\alpha_i : U_i \to V_i$ such that $D_i = \alpha_i^{-1}(D \cap V_i)$ has measure zero in $\mathbb{R}^k$ for each i. (Note that being measure zero in $\mathbb{H}^k$ is the same as that in $\mathbb{R}^k$ (why?).) This definition is well-defined because a change of parametrisation preserves the notion of measure zero (because as one of the HW exercises showed, C^1 maps take measure zero sets to measure zero sets).