MA 200 - Lecture 28

1 Recap

1. Differential forms on manifolds and exterior derivative.
2. Integration of forms on manifolds.
3. Generalised Stokes' theorem and its proof.

2 The generalised Stokes theorem

Using this version of Stokes, we can recover our UM 102 theorems:

1. Green: Let $\Omega \subset \mathbb{R}^2$ be an open set whose topological boundary is a collection of simple closed bounded parametrised smooth curves that are smooth compact 1-manifolds. Then $\bar{\Omega}$ is a smooth manifold-with-boundary (whose boundary is the topological boundary) - HW. Let P, Q be smooth functions on $\bar{\Omega}$. Then $\int_C (Pdx + Qdy) = \int_\Omega d(Pdx + Qdy) = \int_\Omega (Q_x - P_y)dxdy$ provided C is oriented with the restricted orientation, i.e., in the UM 102 way.

2. Stokes: Let $M \subset \mathbb{R}^3$ be a smooth compact oriented surface-with-boundary. Let $\vec{F}$ be a smooth vector field on M. Then let $\omega = F_1dx + F_2dy + F_3dz$. Stokes implies $\int_M d\omega = \int_{\partial M} \omega$. Now suppose α is an orientation-compatible parametrisation of the interior. Then $\alpha^*(d\omega) = \alpha^*((\nabla \times \vec{F})_1 dy \wedge dz + \ldots) = (\nabla \times \vec{F})_1 \circ \alpha(\partial_u \alpha_2 \partial_v \alpha_3 - \partial_u \alpha_3 \partial_v \alpha_2) + \ldots$ which corresponds to $(\nabla \times \vec{F}).d\vec{A}$.

3. Divergence: Let $\Omega \subset \mathbb{R}^3$ be an open set whose topological boundary is a collection of smooth compact surfaces. Then $\bar{\Omega}$ is a smooth compact 3-manifold-with-boundary whose boundary is the topological boundary) - HW. Now let $\vec{F}$ be a smooth vector field on $\bar{\Omega}$. Consider the 2-form $\omega = F_1 dy \wedge dz + \ldots$. Then $d\omega = \nabla.\vec{F} dx \wedge dy \wedge dz$. Thus the generalised Stokes theorem gives us what we want.

3 Poincaré lemma

An open set $U \subset \mathbb{R}^n$ is called star-shaped if there is a point $y \in U$ such that for every point $x \in U$, the line segment $tx + (1-t)y$ (where $0 \leq t \leq 1$) lies entirely within U. For instance, an open ball is star-shaped. Here is an elementary point from vector calculus in $\mathbb{R}^3$: Suppose $U \subset \mathbb{R}^3$ is star-shaped, then whenever $\nabla \times \vec{F} = \vec{0}$, $\vec{F} = \nabla f$. Indeed, one prove it by identifying the "potential function" f by calculating the "work" done by $\vec{F}$. Indeed, define $f(b) = \int_0^1 \vec{F}(tb + (1-t)y).(b-y)dt$. We would like to say that

$$\frac{\partial f}{\partial x} = \int_0^1 \Bigg(F_1(tx + (1-t)y) + t\frac{\partial F_1}{\partial x}(tb + (1-t)y)(b_1 - y_1)$$
$$+t\frac{\partial F_2}{\partial x}(tb + (1-t)y)(b_2 - y_2) + t\frac{\partial F_3}{\partial x}(tb + (1-t)y)(b_3 - y_3)\Bigg)dt. \quad (1)$$

Now we can use the curl condition, the chain rule, and the fundamental theorem of calculus to complete the proof (how?) This "proof" raises a few questions:

1. Why can we differentiate inside the integral sign? (This is a theorem that should have been proven in UM 204.)
2. Does this work for general differential forms? (Yes. This is called Poincaré 's lemma.)

For the first question, here is a theorem.

Theorem 1. *Let $Q \subset \mathbb{R}^n$ be a closed bounded rectangle. Let $f : Q \times [a,b] \to \mathbb{R}$ be a continuous function. Then the function $F(x) = \int_a^b f(x,t)dt$ is continuous on Q. Moreover, if $\frac{\partial f}{\partial x_j}$ is continuous on $Q \times [a,b]$, then $\frac{\partial F}{\partial x_j}$ exists and equals $\int_a^b \frac{\partial f}{\partial x_j}(x,t)dt$.*

Proof. Indeed since f is uniformly continuous, there is δ such that if $\|(x,a)-(y,b)\| < \delta$, then $|f(x,a) - f(y,b)| < \frac{\epsilon}{b-a}$ for all $(x,a),(y,b) \in Q \times [a,b]$. Thus, $|F(x) - F(y)| < \int_a^b |f(x,t) - f(y,t)|dt < \epsilon$ if $\|x - y\| < \delta$. Thus F is continuous.
Wlog $n = 1$ (why?) Let $G(x) = \int_a^b \frac{\partial f}{\partial x}(x,t)dt$ for $x \in [c,d]$. Since we know that the partial of f is continuous, by Fubini and FTC, $\int_c^{x_0} G(x) = \int_c^{x_0}\int_a^b \frac{\partial f}{\partial x}(x,t)dtdx = \int_a^b (f(x_0,t) - f(c,t))dt = F(x_0) - F(c)$. Thus by FTC (we know that G is continuous by the earlier step), $F'(x) = G(x)$. □

Now we can state and prove Poincaré 's lemma.

Theorem 2. *If $U \subset \mathbb{R}^n$ is open and star-shaped, then every closed form on U is exact.*

Proof. Let $\eta = \sum_{i_1<i_2<\ldots}\sum_{a=1}^l (-1)^{a-1}(x_{i_a} - y_{i_a})\int_0^1 t^{l-1}\omega_I(y + t(x-y))dtdx_{i_1} \wedge dx_{i_2} \ldots \wedge dx_{i_{a-1}} \wedge dx_{i_{a+1}} \ldots$. Note that η is smooth by the previous result and induction. We claim that $d\eta = \omega$. Indeed,

$$d\eta = l\int_0^1 t^{l-1}\omega(y + t(x-y))dt + \sum_I \sum_{a=1}^l (-1)^{a-1}\int_0^1 t^l(x_{i_a} - y_{i_a})d\omega_I(y + t(x-y)) \wedge dx_{i_1} \ldots. \quad (2)$$

We claim that the above expression is $\int_0^1 \frac{d(t^l\omega(y+t(x-y)))}{dt}dt = \omega(x)$. Indeed, the first term of the product rule (after differentiating inside the integral sign) matches up. (This is the reason for t^l as opposed to t as in the case of a conservative force.) As for the second term (i.e., $\sum_i \int_0^1 t^l \frac{\partial \omega}{\partial x_i}(y+t(x-y))(x_i-y_i)dt$), it is somewhat complicated. There is a neat trick here. Consider the map $F : U \times [0,1] \to U$ given by $F(x,t) = y + t(x-y)$. Note that $F^*\omega$ is closed. Now

$$F^*\omega = t^l \sum_I \omega_I(F(x,t))dx_{i_1}\ldots$$
$$+\sum_I \sum_a t^{l-1}(-1)^{a-1}(x_{i_a}-y_{i_a})\omega_I(F(x,t))dtdx_{i_1}\wedge\ldots dx_{i_{a-1}}\wedge dx_{i_a}\ldots$$
$$\Rightarrow 0 = dF^*\omega = d_t t^l \sum_I \omega_I(F(x,t))dx_{i_1}\ldots + d_x t^l \sum_I \omega_I(F(x,t))dx_{i_1}\ldots +$$
$$d_t \sum_I \sum_a t^{l-1}(-1)^{a-1}(x_{i_a}-y_{i_a})\omega_I(F(x,t))dtdx_{i_1}\wedge\ldots dx_{i_{a-1}}\wedge dx_{i_a}\ldots$$
$$+d_x \sum_I \sum_a t^{l-1}(-1)^{a-1}(x_{i_a}-y_{i_a})\omega_I(F(x,t))dtdx_{i_1}\wedge\ldots dx_{i_{a-1}}\wedge dx_{i_a}\ldots$$
$$= \frac{\partial(t^l\omega(F(t,x)))}{dt}(-1)^l \wedge dt + 0 + 0$$
$$+(-1)^{l-1}d_x \sum_I \sum_a t^{l-1}(-1)^{a-1}(x_{i_a}-y_{i_a})\omega_I(F(x,t))dx_{i_1}\wedge\ldots dx_{i_{a-1}}\wedge dx_{i_a}\wedge dt. \quad (3)$$

Now integrating with respect to t, we see that $d\eta = \omega$. □