MA 200 - Lecture 14

1 Recap

1. Basic properties of integration.

2. Define measure zero sets and proved a few properties (about unions and so on).

2 Measure zero sets and integrability (and everything else from Munkres)

Here are a few properties:

1. If Q is a rectangle (not a point) in $\mathbb{R}^n$, then the boundary has measure 0 but Q does not: For each boundary face, enlarge it by $\frac{\epsilon}{4^n}$. The sum of the volumes of these enlarged rectangles is less than ϵ and covers the boundary. (These are finitely many rectangles. To get a countable collection, simply choose the other rectangles to be points.)
   As for Q not having measure 0, if it did, cover it with finitely many open rectangles (by compactness and the previous step) whose total volume is less than $v(Q)$. This is a contradiction.

Now we have enough machinery to prove Lebesgue's theorem:

Theorem 1. *Let $Q \subset \mathbb{R}^n$ be a closed rectangle and $f : Q \to \mathbb{R}$ be a bounded function. Let $D \subset Q$ be the set of discontinuities of f. Then f is R.I iff D has measure zero in $\mathbb{R}^n$.*

Proof. Let $|f(x)| \leq M \ \forall\, x \in Q$.

1. If D has measure 0: Roughly speaking, we shall cover D with countably many rectangles of small total volume, and we shall cover the other points by rectangles where $M_R - m_R$ is small. Since Q is compact, only finitely many of all of these rectangles are necessary and using the endpoints of these finitely many rectangles, we shall produce a partition P such that $U(P, f) - L(P, f) < \epsilon$.
   Cover D by open rectangles $Int(Q_1), \ldots$ of total volume less than ϵ' (which we shall see later ought to be chosen to be $\frac{\epsilon}{2M+2v(Q)}$). If a is a point where f is continuous, choose an open rectangle $Int(Q_a)$ containing a such that $|f(x) - f(a)| < \epsilon'$ when $x \in Q_a \cap Q$ (the closed rectangle). Then $Int(Q_i), Int(Q_a)$ cover Q. Since Q is compact, a finite subcollection (that we shall relabel if necessary)

$Int(Q_1), \ldots, Int(Q_k), Int(Q_{a_1}), \ldots Int(Q_{a_l})$ cover Q (these need not cover all the discontinuities or all continuities). Replace Q_{a_j} and Q_i by their intersections with Q. Take a partition P given by the endpoints of each component interval of Q_i and Q_{a_j}. Then each closed rectangle is a union of sub-rectangles of this partition. Now every sub-rectangle R is either in Q_i or in Q_{a_j}. If it is in the former, then $(M_R - m_R)v(R) \leq 2Mv(R)$ and if it is in the latter, $(M_R - m_R)v(R) < 2\epsilon' v(R)$. Thus, $U(P, f) - L(P, f) \leq 2M\epsilon' + 2\epsilon' v(Q) = (2M + 2v(Q))\epsilon' = \epsilon$ if $\epsilon' = \frac{\epsilon}{2M+2v(Q)}$.

2. If f is R.I: The idea is that if $U(P, f) - L(P, f) = \sum_R (M_R - m_R)v(R) < \epsilon$, then the rectangles that cover discontinuities ought to have a relatively large $M_R - v_R$. (If so, then their total volume is small and we are done.) This is not quite true (because $M_R - v_R$ can still be $O(\epsilon)$ for some such rectangles for instance). So we shall quantify the amount of discontinuity at every point:
   The oscillation of f at a point $a \in Q$ is defined as $o(f, a) = \inf_{\delta>0}(M_\delta f - m_\delta f)$ where $M_\delta(f) = \sup_{|x-a|<\delta} f(x)$ and $m_\delta(f) = \inf_{|x-a|<\delta} f(x)$. It is easy to see that $o(f, a) = 0$ iff f is continuous at a (why?).
   Let D_m be the set of all $x \in Q$ such that $o(f, x) \geq \frac{1}{m}$. Clearly, $D = \cup_{m\in\mathbb{Z}_{>0}} D_m$. If prove that each D_m has measure zero, we are done. Indeed, since f is R.I, given $\epsilon > 0$, there is a partition P such that $\sum_R (M_R - m_R)v(R) < \frac{\epsilon}{2m}$. Now every point in D_m is either in the boundary of a closed rectangle R or in the interior of one. The collection of all the ones in the boundary have measure 0 (because the boundaries of each of these countably many rectangles has measure 0) and hence can be covered by countably many closed rectangles of total measure $\frac{\epsilon}{2}$. Suppose $Int(R_{i_1}), \ldots, Int(R_{i_k})$ are the rectangles containing all the other points of D_m. Then $\frac{1}{m}\sum_k v(R_{i_k}) \sum_k (M_{R_{i_k}} - m_(R_{i_k}))v(R_{i_k}) < \frac{\epsilon}{2m}$ and hence $\sum_k v(R_{i_k}) < \frac{\epsilon}{2}$. Thus we are done.

□

As a consequence, piecewise continuous functions (with finitely many pieces) are R.I. Moreover,

Theorem 2. *Assume f is integrable over Q.*

1. *If $f = 0$ everywhere except on a set E of measure 0 (vanishes "almost everywhere"), then $\int_Q f = 0$.*
2. *If $f \geq 0$ and $\int_Q f = 0$, then f vanishes almost everywhere.*

Proof. 1. Let P be a partition. If R is any subrectangle, R is not contained in E and hence has a point where f vanishes. Thus, $m_R \leq 0$ and $M_R \geq 0$. Thus, the L.I is ≤ 0 and the $U.I \geq 0$. Since they coincide, the integral is 0.

2. Since the integral exists, f is continuous almost everywhere. At all these points of continuity, we claim that $f = 0$. Indeed, if $f(a) > 0$ at some a, then $f(x) \geq c > 0$ for all $|x - a| < \delta$. Now, choose a partition P of mesh $< \delta$. For any rectangle containing a, $m_R(f) \geq c$. Thus, $L(f, P) \geq v(R_0)c > 0$. But $L(f, P) \leq \int_Q f = 0$.

□

3 Evaluation of integrals

Recall the fundamental theorem of calculus.

Theorem 3. *If $f : [a, b] \to \mathbb{R}$ is continuous,*

1. *then $F(x) = \int_a^x f(t)dt$ is differentiable and $F'(x) = f(x)\ \forall\ x \in [a, b]$.*
2. *and if $F(x)$ is an antiderivative of f on $[a, b]$, i.e., $F'(x) = f(x)\ \forall\ x \in [a, b]$, then $F(b) - F(a) = \int_a^b f(x)dx$.*

What about multiple integrals? How can we evaluate them?

Theorem 4 (Fubini). *Let $Q = A \times B$ where $A \subset \mathbb{R}^k$ and $B \subset \mathbb{R}^n$ are rectangles. Let $f : Q \to \mathbb{R}$ be a bounded function. If f is R.I over Q, then $\underline{\int_{y\in B}} f(x, y)dy$ and $\overline{\int_{y\in B}} f(x, y)dy$ are integrable over A and $\int_Q f = \int_{x\in A} \underline{\int_{y\in B}} f(x, y)dydx = \int_{x\in A} \overline{\int_{y\in B}} f(x, y)dydx$. (In particular, if f is continuous, then the iterated integral exists and equals the multiple integral in any order.)*

Proof. An easy comparison of sums over partitions (using of course, $v(R_A \times R_B) = v(R_A)v(R_B)$, and the finite-sum version of Fubini). □

Now we can integrate say, $x^2y + 4x^3z^2$ over a rectangle explicitly.