MA 200 - Lecture 7

1 Recap

1. Inverse function theorem

2 Inverse function theorem

Proof. We first prove for $r = 1$:

1. f is locally $1-1$.
2. The image of a neighbourhood of f is open.
3. f^{-1} is continuous: Note that on U, Df is invertible at every point $x \in U$. Hence, applying the previous step to each x, we conclude that the image of every open subset of U is open (how?) Hence $f^{-1} : f(U) \to U$ is open and hence f^{-1} is continuous (why?)
4. f^{-1} is C^1: Firstly, f^{-1} is differentiable with derivative $[Df_{f^{-1}(b)}]^{-1}$. Indeed, $f^{-1}(b+h) - f^{-1}(b) = k$. Thus $b + h = f(f^{-1}(b) + k)$. Thus, for each component $h_i = \langle \nabla f_i(\theta_i), k\rangle$ (by MVT). In other words, $h = Bk$ where the i^{th} row of B is $\nabla f_i(\theta_i)$ (where $\theta_i \to f^{-1}(b)$ as $k \to 0$. Thus by continuity of f^{-1}, as $h \to 0$, $B \to Df_{f^{-1}(b)}$). This implies that $\frac{\|k-[Df_{f^{-1}(b)}]^{-1}h\|}{\|h\|} = \frac{\|B^{-1}h-[Df_{f^{-1}(b)}]^{-1}h\|}{\|h\|} \leq \|B^{-1} - [Df_{f^{-1}(b)}]^{-1}\| \to 0$ as $h \to 0$.
   Since $Df^{-1}(y) = [Df_{f^{-1}(y)}]^{-1}$, by properties of continuity f is C^1.

Now we shall prove it for general r: Assume it is true for $1, 2, \ldots, r-1$. Then $D(f^{-1})_y = [Df_{f^{-1}(y)}]^{-1}$. This is C^{r-1} (why?) Hence, f^{-1} is C^r. $\square$

The IFT motivates a definition: Let $U, V \subset \mathbb{R}^n$ be open subsets. A function $f : U \to V$ that is $1-1$, onto, C^r, and whose inverse is also C^r (where $1 \leq r \leq \infty$) is called a C^r-diffeomorphism between U and V. (If $r = 0$, it is called a homeomorphism.)
So IFT states that if Df_a is invertible, then f is a local C^r diffeomorphism. Now $(r,\theta) \in (0,\infty) \times (0, 2\pi) \to (r\cos(\theta), r\sin(\theta)) \in \mathbb{R}^2 - \{(x,y)|x = 0, y \geq 0\}$ is an example of a C^∞ diffeomorphism. These are the famous polar coordinates. The terminology raises a question? Are there other coordinates? What are coordinates after all? (They are called "frames of reference" in physics) We will answer these questions some time but basically, diffeomorphisms are used to get new coordinate systems. (Einstein wanted the laws of physics to remain invariant under change of frame, i.e., under

diffeomorphisms. Newton merely wanted them to be invariant under *linear* changes with time remaining unchanged, i.e., inertial frames.)