MA 200 - Lecture 21

1 Recap

1. Defined manifolds-with-boundary, gave an example (unit disc), interior and boundary points.
2. Defined integrals of continuous functions over compact manifolds-with-boundary.
3. Defined measure zero on manifolds.

2 Manifolds-with-boundary

Theorem 1. *Let $M \subset \mathbb{R}^n$ be a compact C^r k-manifold with or without boundary. Let $f : M \to \mathbb{R}$ be a continuous function. Suppose $\alpha_i : A_i \to M_i$ are (finitely many) coordinate parametrisations (where A_i are open subsets of either $\mathbb{R}^k$ or $\mathbb{H}^k$), and M is the disjoint union of the open sets M_i and a set K of measure zero in M. Then*

$$\int f dV = \sum_i Improper \int_{Int(A_i)} (f \circ \alpha_i)\sqrt{\det(D\alpha_i^T D\alpha_i)}. \tag{1}$$

Proof. Since both sides are linear in f, WLOG, we can assume that the support of f is contained in a bounded single coordinate parametrisation $\alpha : U \to V$. Then $\int_M f dV = \int_U (f \circ \alpha)\sqrt{\det(D\alpha^T D\alpha)}$. Let $W_i = \alpha^{-1}(M_i \cap V)$ and $L = \alpha^{-1}(K \cap V)$. Then W_i are open sets in $\mathbb{R}^k$ or $\mathbb{H}^k$ and L has measure zero in $\mathbb{R}^k$. Moreover, U is the disjoint union of W_i and L. Note that $Improper \sum_i \int_{Int(W_i)} (f \circ \alpha)\sqrt{\det(D\alpha^T D\alpha)}$ exist as usual integrals (why?) and by additivity and the fact that L has measure zero, this sum is $\int_{Int(U)} (f \circ \alpha)\sqrt{\det(D\alpha^T D\alpha)} = \int_M f dV$. Now we claim that the integral of $F = (f \circ \alpha)\sqrt{\det(D\alpha^T D\alpha)}$ over $Int(W_i)$ is the integral of $F_i = (f \circ \alpha_i)\sqrt{\det(D\alpha_i^T D\alpha_i)}$ over $Int(A_i)$. By the change of variables formula, the integral of F over $Int(W_i)$ is the improper integral of F_i over the interior of $B_i = \alpha_i^{-1}(M_i \cap V)$. This follows from the additivity property for improper integrals. $\square$

As a consequence, we can calculate the surface area of a sphere using the usual parametrisation by polar coordinates.

3 A sketch of proof of Green's theorem

A version of Green: Let $\Omega \subset \mathbb{R}^2$ be a compact set, $f : \Omega \to \mathbb{R}$ a smooth function, $f = 0$ is a regular level set, and $f \geq 0$ is Ω. Suppose $\partial\Omega$ can be parametrised-upto-measure zero by a single patch $\gamma : (0,1) \to \mathbb{R}^2$ such that $\nabla f \times \gamma'$ points in the $\hat{k}$ direction throughout $\gamma(0,1)$. Let $P, Q : \Omega \to \mathbb{R}$ be smooth functions. Then $\int_0^1((P \circ \gamma)\gamma_1' + (Q \circ \gamma)\gamma_2')dt = \int_{Int(\Omega)}(Q_x - P_y)$.

Proof. Cover the boundary by boundary coordinate patches U_i (of the form $\alpha^{-1}(x,y) \to (x,f)$ (when $f_y > 0$) or $(-y,f)$ (when $f_x > 0$), etc). Note that these changes of variables have positive Jacobians) and the interior by the usual patch V. Choose a partition-of-unity ρ_j subordinate to this cover. By linearity, we can assume WLog that P, Q have supports in one of these coordinate patches. If that patch is V, then the RHS is zero (because it is trivial to prove Green for rectangles) and so is the LHS. If it is one of the U_i, then by change of variables, we can reduce to a rectangle and be done. Now the fact that the integral can be calculated by only one patch $\gamma(0,1)$ follows from the measure zero business. The key point is that the direction of γ' is the right one for the Green theorem over a rectangle. □

4 Orientability of manifolds

In the above sketch of proof, it appears crucial that the integral be such that it changes by the Jacobian upon change of variables and that we have successfully covered the manifold-with-boundary by coordinate patches where the change of patch Jacobian is *positive*. We generalise the latter property into a definition as follows. (The former property will also have to be generalised to higher dimensions.)
Let $g : A \subset \mathbb{R}^k \to B \subset \mathbb{R}^k$ be a diffeo. It is said to be orientation-preserving if $\det(Dg) > 0$ everywhere. It is said to be orientation-reversing if $\det(Dg) < 0$ everywhere. (Note that if A is connected, then only one of these possibilities occurs.)
Let $M \subset \mathbb{R}^n$ be a k-dimensional manifold with nonempty or without boundary. Given two coordinate parametrisations $\alpha_i : U_i \to V_i$, we say that they are orientation-compatible with each other if the transition functions $\alpha_i \circ \alpha_j^{-1}$ are orientation-preserving. If $k \geq 2$, and M can be covered with coordinate patches that are mutually orientation-compatible with each other, then M is said to be orientable and the given collection of compatible coordinate patches, augmented with all possible coordinate patches that are compatible with the given ones, is said to be an orientation of M.
Given a parametrisation α, we can *reverse* its orientation: $\beta = (-\alpha_1, \alpha_2, \ldots)$. Now $\alpha \circ \beta^{-1}(x_1, x_2, \ldots, x_k) = (-x_1, \ldots, x_k)$, which is an orientation-reversing diffeo. Thus, given an oriented manifold, we can reverse all the orientation-compatible parametrisations and produce *another* orientation called the opposite orientation.

1. An oriented 1-manifold M (according to the $k \geq 2$ definition) has a C^r-varying unit-speed tangent vector field, i.e., a function $T : M \to \mathbb{R}^n$ such that $T(p) \in T_pM \; \forall \, p \in M$ (note that T_pM is the span of $\frac{\partial \alpha}{\partial x_i}(p)$ for an parametrisation), for any parametrisation α, $T \circ \alpha$ is a C^r function, and $\|T(p)\| = 1 \; \forall \, p$. Indeed, cover M with orientation-compatible coordinate parametrisations. Then define $T(p) = \frac{\alpha_i'(t)}{\|\alpha_i'(t)\|}$.

This definition is independent of i.

However, the converse is not true in the case of $k = 1$. The problem is that $[0, 1] \subset \mathbb{R}$ is not "orientable": Indeed, if there is such a $T : [0, 1] \to \mathbb{R}$ that is compatible with orientation-preserving charts, then suppose the usual interior coordinate chart is orientation-compatible on $\mathbb{R}$. The boundary charts near 1 and 0 "point" in opposite directions and hence we have a problem. Thus we *define* orientation for 1-manifolds to simply be the existence of a C^r-varying unit-speed tangent vector field (the opposite orientation comes from simply $-T$).

2. Here is a lemma that produces several examples: Let M be an $n - 1$-dimensional (where $n - 1 \geq 2$) manifold in $\mathbb{R}^n$. A C^r-varying unit normal vector field on M is a function $n : M \to \mathbb{R}^n$ such that for any coordinate patch α, $n \circ \alpha$ is C^r (by the chain rule, this can be accomplished by making sure such is the case for *some* collection of patches that cover M), $n(p) \perp T_pM \ \forall \ p$. Now M is orientable iff it has a C^r-varying unit normal vector field:

3. A Möbius strip is not orientable (a challenging exercise).